16. Differentiation

Definition 115 *Let* $(\Omega, \mathcal{T})$ *be a topological space. A map* $f : \Omega \to \bar{\mathbf{R}}$ *is said to be* **lower-semi-continuous** *(l.s.c), if and only if:*

$$\forall \lambda \in \mathbf{R}\ ,\ \{\lambda < f\} \text{ is open}$$

We say that f *is* **upper-semi-continuous** *(u.s.c), if and only if:*

$$\forall \lambda \in \mathbf{R}\ ,\ \{f < \lambda\} \text{ is open}$$

Exercise 1. Let $f : \Omega \to \bar{\mathbf{R}}$ be a map, where Ω is a topological space.

1. Show that f is l.s.c if and only if $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.
2. Show that f is u.s.c if and only if $\{f < \lambda\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.
3. Show that every open set U in $\bar{\mathbf{R}}$ can be written:

$$U = V^+ \cup V^- \cup \bigcup_{i \in I}]\alpha_i, \beta_i[$$

for some index set I, $\alpha_i, \beta_i \in \mathbf{R}$, $V^+ = \emptyset$ or $V^+ =]\alpha, +\infty]$, $(\alpha \in \mathbf{R})$ and $V^- = \emptyset$ or $V^- = [-\infty, \beta[$, $(\beta \in \mathbf{R})$.

4. Show that f is continuous if and only if it is both l.s.c and u.s.c.

5. Let $u : \Omega \to \mathbf{R}$ and $v : \Omega \to \bar{\mathbf{R}}$. Let $\lambda \in \mathbf{R}$. Show that:

$$\{\lambda < u + v\} = \bigcup_{\substack{(\lambda_1, \lambda_2) \in \mathbf{R}^2 \\ \lambda_1 + \lambda_2 = \lambda}} \{\lambda_1 < u\} \cap \{\lambda_2 < v\}$$

6. Show that if both u and v are l.s.c, then $u + v$ is also l.s.c.

7. Show that if both u and v are u.s.c, then $u + v$ is also u.s.c.

8. Show that if f is l.s.c, then αf is l.s.c, for all $\alpha \in \mathbf{R}^+$.

9. Show that if f is u.s.c, then αf is u.s.c, for all $\alpha \in \mathbf{R}^+$.

10. Show that if f is l.s.c, then $-f$ is u.s.c.

11. Show that if f is u.s.c, then $-f$ is l.s.c.

12. Show that if V is open in Ω, then $f = 1_V$ is l.s.c.

13. Show that if F is closed in Ω, then $f = 1_F$ is u.s.c.

Exercise 2. Let $(f_i)_{i\in I}$ be an a arbitrary family of maps $f_i : \Omega \to \bar{\mathbf{R}}$, defined on a topological space Ω.

1. Show that if all f_i's are l.s.c, then $f = \sup_{i\in I} f_i$ is l.s.c.

2. Show that if all f_i's are u.s.c, then $f = \inf_{i\in I} f_i$ is u.s.c.

Exercise 3. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let f be an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $f \geq 0$.

1. Let $(s_n)_{n\geq 1}$ be a sequence of simple functions on $(\Omega, \mathcal{B}(\Omega))$ such that $s_n \uparrow f$. Define $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for all $n \geq 2$. Show that t_n is a simple function on $(\Omega, \mathcal{B}(\Omega))$, for all $n \geq 1$.

2. Show that f can be written as:

$$f = \sum_{n=1}^{+\infty} \alpha_n 1_{A_n}$$

   where $\alpha_n \in \mathbf{R}^+ \setminus \{0\}$ and $A_n \in \mathcal{B}(\Omega)$, for all $n \geq 1$.

3. Show that $\mu(A_n) < +\infty$, for all $n \geq 1$.

4. Show that there exist K_n compact and V_n open in Ω such that:

$$K_n \subseteq A_n \subseteq V_n \quad , \quad \mu(V_n \setminus K_n) \leq \frac{\epsilon}{\alpha_n 2^{n+1}}$$

   for all $\epsilon > 0$ and $n \geq 1$.

5. Show the existence of $N \geq 1$ such that:

$$\sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}$$

6. Define $u = \sum_{n=1}^{N} \alpha_n 1_{K_n}$. Show that u is u.s.c.

7. Define $v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n}$. Show that v is l.s.c.

8. Show that we have $0 \leq u \leq f \leq v$.

9. Show that we have:

$$v = u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n}$$

10. Show that $\int v d\mu \leq \int u d\mu + \epsilon < +\infty$.

11. Show that $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

12. Explain why v may fail to be in $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

13. Show that v is μ-a.s. equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

14. Show that $\int(v-u)d\mu \leq \epsilon$.

15. Prove the following:

Theorem 94 (Vitali-Caratheodory) *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ and f be an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$. Then, for all $\epsilon > 0$, there exist measurable maps $u, v : \Omega \to \bar{\mathbf{R}}$, which are μ-a.s. equal to elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $u \leq f \leq v$, u is u.s.c, v is l.s.c, and furthermore:*

$$\int(v-u)d\mu \leq \epsilon$$

Definition 116 *Let* $(\Omega, \mathcal{T})$ *be a topological space. We say that* $(\Omega, \mathcal{T})$ *is* **connected**, *if and only if the only subsets of* Ω *which are both open and closed are* Ω *and* $\emptyset$.

EXERCISE 4. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that $(\Omega, \mathcal{T})$ is connected if and only if whenever $\Omega = A \uplus B$ where A, B are disjoint open sets, we have $A = \emptyset$ or $B = \emptyset$.

2. Show that $(\Omega, \mathcal{T})$ is connected if and only if whenever $\Omega = A \uplus B$ where A, B are disjoint closed sets, we have $A = \emptyset$ or $B = \emptyset$.

Definition 117 *Let* $(\Omega, \mathcal{T})$ *be a topological space, and* $A \subseteq \Omega$. *We say that* A *is a* **connected subset** *of* Ω, *if and only if the induced topological space* $(A, \mathcal{T}_{|A})$ *is connected.*

EXERCISE 5. Let A be open and closed in $\mathbf{R}$, with $A \neq \emptyset$ and $A^c \neq \emptyset$.

1. Let $x \in A^c$. Show that $A \cap [x, +\infty[$ or $A \cap]-\infty, x]$ is non-empty.

2. Suppose $B = A \cap [x, +\infty[\neq \emptyset$. Show that B is closed and that we have $B = A \cap]x, +\infty[$. Conclude that B is also open.

3. Let $b = \inf B$. Show that $b \in B$ (and in particular $b \in \mathbf{R}$).

4. Show the existence of $\epsilon > 0$ such that $]b - \epsilon, b + \epsilon[\subseteq B$.

5. Conclude with the following:

Theorem 95 *The topological space* $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ *is connected.*

EXERCISE 6. Let $(\Omega, \mathcal{T})$ be a topological space and $A \subseteq \Omega$ be a connected subset of Ω. Let B be a subset of Ω such that $A \subseteq B \subseteq \bar{A}$. We assume that $B = V_1 \uplus V_2$ where V_1, V_2 are disjoint open sets in B.

1. Show there is U_1, U_2 open in Ω, with $V_1 = B \cap U_1$, $V_2 = B \cap U_2$.

2. Show that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$.

3. Suppose that $A \cap U_1 = \emptyset$. Show that $\bar{A} \subseteq U_1^c$.

4. Show then that $V_1 = B \cap U_1 = \emptyset$.

5. Conclude that B and $\bar{A}$ are both connected subsets of Ω.

Exercise 7. Prove the following:

Theorem 96 *Let $(\Omega, \mathcal{T})$, $(\Omega', \mathcal{T}')$ be two topological spaces, and f be a continuous map, $f : \Omega \to \Omega'$. If $(\Omega, \mathcal{T})$ is connected, then $f(\Omega)$ is a connected subset of Ω'.*

Definition 118 *Let $A \subseteq \bar{\mathbf{R}}$. We say that A is an* **interval***, if and only if for all $x, y \in A$ with $x \leq y$, we have $[x, y] \subseteq A$, where:*

$$[x, y] \stackrel{\triangle}{=} \{z \in \bar{\mathbf{R}} \ : \ x \leq z \leq y\}$$

Exercise 8. Let $A \subseteq \bar{\mathbf{R}}$.

1. If A is an interval, and $\alpha = \inf A$, $\beta = \sup A$, show that:
$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$$
2. Show that A is an interval if and only if, it is of the form $[\alpha, \beta]$, $[\alpha, \beta[$, $]\alpha, \beta]$ or $]\alpha, \beta[$, for some $\alpha, \beta \in \bar{\mathbf{R}}$.
3. Show that an interval of the form $]-\infty, \alpha[$, where $\alpha \in \mathbf{R}$, is homeomorphic to $]-1, \alpha'[$, for some $\alpha' \in \mathbf{R}$.
4. Show that an interval of the form $]\alpha, +\infty[$, where $\alpha \in \mathbf{R}$, is homeomorphic to $]\alpha', 1[$, for some $\alpha' \in \mathbf{R}$.
5. Show that an interval of the form $]\alpha, \beta[$, where $\alpha, \beta \in \mathbf{R}$ and $\alpha < \beta$, is homeomorphic to $]-1, 1[$.
6. Show that $]-1, 1[$ is homeomorphic to $\mathbf{R}$.
7. Show an non-empty open interval in $\mathbf{R}$, is homeomorphic to $\mathbf{R}$.

8. Show that an open interval in $\mathbf{R}$, is a connected subset of $\mathbf{R}$.

9. Show that an interval in $\mathbf{R}$, is a connected subset of $\mathbf{R}$.

Exercise 9. Let $A \subseteq \mathbf{R}$ be a non-empty connected subset of $\mathbf{R}$, and $\alpha = \inf A$, $\beta = \sup A$. We assume there exists $x_0 \in A^c \cap]\alpha, \beta[$.

1. Show that $A \cap]x_0, +\infty[$ or $A \cap] - \infty, x_0[$ is empty.

2. Show that $A \cap]x_0, +\infty[= \emptyset$ leads to a contradiction.

3. Show that $]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$.

4. Show the following:

Theorem 97 *For all $A \subseteq \mathbf{R}$, A is a connected subset of $\mathbf{R}$, if and only if A is an interval.*

Exercise 10. Prove the following:

Theorem 98 *Let $f : \Omega \to \mathbf{R}$ be a continuous map, where $(\Omega, \mathcal{T})$ is a connected topological space. Let $a, b \in \Omega$ such that $f(a) \leq f(b)$. Then, for all $z \in [f(a), f(b)]$, there exists $x \in \Omega$ such that $z = f(x)$.*

Exercise 11. Let $a, b \in \mathbf{R}$, $a < b$, and $f : [a, b] \to \mathbf{R}$ be a map such that $f'(x)$ exists for all $x \in [a, b]$.

1. Show that $f' : ([a, b], \mathcal{B}([a, b])) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.
2. Show that $f' \in L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$ is equivalent to:
$$\int_a^b |f'(t)|dt < +\infty$$
3. We assume from now on that $f' \in L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$. Given $\epsilon > 0$, show the existence of $g : [a, b] \to \bar{\mathbf{R}}$, almost surely equal

to an element of $L^1_{\mathbf{R}}([a,b],\mathcal{B}([a,b]),dx)$, such that $f' \leq g$ and g is l.s.c, with:

$$\int_a^b g(t)dt \leq \int_a^b f'(t)dt + \epsilon$$

4. By considering $g+\alpha$ for some $\alpha > 0$, show that without loss of generality, we can assume that $f' < g$ with the above inequality still holding.

5. We define the complex measure $\nu = \int g dx \in M^1([a,b],\mathcal{B}([a,b]))$. Show that:

$$\forall \epsilon' > 0 \ , \ \exists \delta > 0 \ , \ \forall E \in \mathcal{B}([a,b]) \ , \ dx(E) \leq \delta \ \Rightarrow \ |\nu(E)| < \epsilon'$$

6. For all $\eta > 0$ and $x \in [a,b]$, we define:

$$F_\eta(x) \stackrel{\triangle}{=} \int_a^x g(t)dt - f(x) + f(a) + \eta(x-a)$$

Show that $F_\eta : [a,b] \to \mathbf{R}$ is a continuous map.

7. η being fixed, let $x = \sup F_\eta^{-1}(\{0\})$. Show that $x \in [a, b]$ and $F_\eta(x) = 0$.

8. We assume that $x \in [a, b[$. Show the existence of $\delta > 0$ such that for all $t \in]x, x + \delta[\cap[a, b]$, we have:

$$f'(x) < g(t) \quad \text{and} \quad \frac{f(t) - f(x)}{t - x} < f'(x) + \eta$$

9. Show that for all $t \in]x, x+\delta[\cap[a, b]$, we have $F_\eta(t) > F_\eta(x) = 0$.

10. Show that there exists t_0 such that $x < t_0 < b$ and $F_\eta(t_0) > 0$.

11. Show that $F_\eta(b) < 0$ leads to a contradiction.

12. Conclude that $F_\eta(b) \geq 0$, even if $x = b$.

13. Show that $f(b) - f(a) \leq \int_a^b f'(t)dt$, and conclude:

Theorem 99 (Fundamental Calculus) *Let $a, b \in \mathbf{R}$, $a < b$, and $f : [a, b] \to \mathbf{R}$ be a map which is differentiable at every point of $[a, b]$, and such that:*

$$\int_a^b |f'(t)|dt < +\infty$$

Then, we have:

$$f(b) - f(a) = \int_a^b f'(t)dt$$

EXERCISE 12. Let $\alpha > 0$, and $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ defined by $k_\alpha(x) = \alpha x$.

1. Show that $k_\alpha : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.
2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:

$$dx(\{k_\alpha \in B\}) = \frac{1}{\alpha^n} dx(B)$$

3. Show that for all $\epsilon > 0$ and $x \in \mathbf{R}^n$:

$$dx(B(x, \epsilon)) = \epsilon^n dx(B(0, 1))$$

Definition 119 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, with total variation $|\mu|$. We call* **maximal function** *of μ, the map $M\mu : \mathbf{R}^n \to [0, +\infty]$, defined by:*

$$\forall x \in \mathbf{R}^n \ , \ (M\mu)(x) \stackrel{\triangle}{=} \sup_{\epsilon > 0} \frac{|\mu|(B(x,\epsilon))}{dx(B(x,\epsilon))}$$

where $B(x, \epsilon)$ is the open ball in $\mathbf{R}^n$, of center x and radius ϵ, with respect to the usual metric of $\mathbf{R}^n$.

Exercise 13. Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

1. Let $\lambda \in \mathbf{R}$. Show that if $\lambda < 0$, then $\{\lambda < M\mu\} = \mathbf{R}^n$.
2. Show that if $\lambda = 0$, then $\{\lambda < M\mu\} = \mathbf{R}^n$ if $\mu \neq 0$, and $\{\lambda < M\mu\}$ is the empty set if $\mu = 0$.
3. Suppose $\lambda > 0$. Let $x \in \{\lambda < M\mu\}$. Show the existence of $\epsilon > 0$ such that $|\mu|(B(x, \epsilon)) = t dx(B(x, \epsilon))$, for some $t > \lambda$.

4. Show the existence of $\delta > 0$ such that $(\epsilon+\delta)^n < \epsilon^n t/\lambda$.

5. Show that if $y \in B(x,\delta)$, then $B(x,\epsilon) \subseteq B(y,\epsilon+\delta)$.

6. Show that if $y \in B(x,\delta)$, then:

$$|\mu|(B(y,\epsilon+\delta)) \geq \frac{\epsilon^n t}{(\epsilon+\delta)^n} dx(B(y,\epsilon+\delta)) > \lambda dx(B(y,\epsilon+\delta))$$

7. Conclude that $B(x,\delta) \subseteq \{\lambda < M\mu\}$, and that the maximal function $M\mu : \mathbf{R}^n \to [0,+\infty]$ is l.s.c, and therefore measurable.

Exercise 14. Let $B_i = B(x_i,\epsilon_i)$, $i = 1,\ldots,N$, $N \geq 1$, be a finite collection of open balls in $\mathbf{R}^n$. Assume without loss of generality that $\epsilon_N \leq \ldots \leq \epsilon_1$. We define a sequence (J_k) of sets by $J_0 = \{1,\ldots,N\}$ and for all $k \geq 1$:

$$J_k \triangleq \begin{cases} J_{k-1} \cap \{j : \ j > i_k \ , \ B_j \cap B_{i_k} = \emptyset\} & \text{if } J_{k-1} \neq \emptyset \\ \emptyset & \text{if } J_{k-1} = \emptyset \end{cases}$$

where we have put $i_k = \min J_{k-1}$, whenever $J_{k-1} \neq \emptyset$.

1. Show that if $J_{k-1} \neq \emptyset$ then $J_k \subset J_{k-1}$ (strict inclusion), $k \geq 1$.
2. Let $p = \min\{k \geq 1 : \ J_k = \emptyset\}$. Show that p is well-defined.
3. Let $S = \{i_1, \ldots, i_p\}$. Explain why S is well defined.
4. Suppose that $1 \leq k < k' \leq p$. Show that $i_{k'} \in J_k$.
5. Show that $(B_i)_{i \in S}$ is a family of pairwise disjoint open balls.
6. Let $i \in \{1, \ldots, N\} \setminus S$, and define k_0 to be the minimum of the set $\{k \in \mathbf{N}_p : \ i \notin J_k\}$. Explain why k_0 is well-defined.
7. Show that $i \in J_{k_0 - 1}$ and $i_{k_0} \leq i$.
8. Show that $B_i \cap B_{i_{k_0}} \neq \emptyset$.
9. Show that $B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$.

10. Conclude that there exists a subset S of $\{1, \ldots, N\}$ such that $(B_i)_{i\in S}$ is a family of pairwise disjoint balls, and:

$$\bigcup_{i=1}^{N} B(x_i, \epsilon_i) \subseteq \bigcup_{i\in S} B(x_i, 3\epsilon_i)$$

11. Show that:

$$dx\left(\bigcup_{i=1}^{N} B(x_i, \epsilon_i)\right) \leq 3^n \sum_{i\in S} dx(B(x_i, \epsilon_i))$$

EXERCISE 15. Let μ be a complex measure on $\mathbf{R}^n$. Let $\lambda > 0$ and K be a non-empty compact subset of $\{\lambda < M\mu\}$.

1. Show that K can be covered by a finite collection $B_i = B(x_i, \epsilon_i)$, $i = 1, \ldots, N$ of open balls, such that:

$$\forall i = 1, \ldots, N\ ,\ \lambda dx(B_i) < |\mu|(B_i)$$

2. Show the existence of $S \subseteq \{1, \ldots, N\}$ such that:

$$dx(K) \leq 3^n \lambda^{-1} |\mu| \left(\bigcup_{i \in S} B(x_i, \epsilon_i) \right)$$

3. Show that $dx(K) \leq 3^n \lambda^{-1} \|\mu\|$

4. Conclude with the following:

Theorem 100 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, with maximal function $M\mu$. Then, for all $\lambda \in \mathbf{R}^+ \setminus \{0\}$, we have:*

$$dx(\{\lambda < M\mu\}) \leq 3^n \lambda^{-1} \|\mu\|$$

Definition 120 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, and μ be the complex measure $\mu = \int f dx$ on $\mathbf{R}^n$, $n \geq 1$. We call* **maximal function** *of f, denoted Mf, the maximal function $M\mu$ of μ.*

Exercise 16. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$.

1. Show that for all $x \in \mathbf{R}^n$:

$$(Mf)(x) = \sup_{\epsilon > 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f| dx$$

2. Show that for all $\lambda > 0$, $dx(\{\lambda < Mf\}) \leq 3^n \lambda^{-1} \|f\|_1$.

Definition 121 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. We say that $x \in \mathbf{R}^n$ is a* **Lebesgue point** *of f, if and only if we have:*

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

Exercise 17. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$.

1. Show that if f is continuous at $x \in \mathbf{R}^n$, then x is a Lebesgue point of f.

2. Show that if $x \in \mathbf{R}^n$ is a Lebesgue point of f, then:

$$f(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} f(y)dy$$

Exercise 18. Let $n \geq 1$ and $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. For all $\epsilon > 0$ and $x \in \mathbf{R}^n$, we define:

$$(T_\epsilon f)(x) \stackrel{\triangle}{=} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)|dy$$

and we put, for all $x \in \mathbf{R}^n$:

$$(Tf)(x) \stackrel{\triangle}{=} \limsup_{\epsilon \downarrow\downarrow 0} (T_\epsilon f)(x) \stackrel{\triangle}{=} \inf_{\epsilon > 0} \sup_{u \in]0,\epsilon[} (T_u f)(x)$$

1. Given $\eta > 0$, show the existence of $g \in C^c_{\mathbf{C}}(\mathbf{R}^n)$ such that:

$$\|f - g\|_1 \leq \eta$$

2. Let $h = f - g$. Show that for all $\epsilon > 0$ and $x \in \mathbf{R}^n$:
$$(T_\epsilon h)(x) \leq \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |h| dx + |h(x)|$$

3. Show that $Th \leq Mh + |h|$.

4. Show that for all $\epsilon > 0$, we have $T_\epsilon f \leq T_\epsilon g + T_\epsilon h$.

5. Show that $Tf \leq Tg + Th$.

6. Using the continuity of g, show that $Tg = 0$.

7. Show that $Tf \leq Mh + |h|$.

8. Show that for all $\alpha > 0$, $\{2\alpha < Tf\} \subseteq \{\alpha < Mh\} \cup \{\alpha < |h|\}$.

9. Show that $dx(\{\alpha < |h|\}) \leq \alpha^{-1} \|h\|_1$.

10. Conclude that for all $\alpha > 0$ and $\eta > 0$, there is $N_{\alpha,\eta} \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_{\alpha,\eta}$ and $dx(N_{\alpha,\eta}) \leq \eta$.

11. Show that for all $\alpha > 0$, there exists $N_\alpha \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_\alpha$ and $dx(N_\alpha) = 0$.

12. Show there is $N \in \mathcal{B}(\mathbf{R}^n)$, $dx(N) = 0$, such that $\{Tf > 0\} \subseteq N$.

13. Conclude that $Tf = 0$, $dx-$a.s.

14. Conclude with the following:

Theorem 101 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. Then, dx-almost surely, any $x \in \mathbf{R}^n$ is a Lebesgue points of f, i.e.*

$$dx\text{-}a.s. \ , \ \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

Exercise 19. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \in \mathcal{F}$. We define $\mathcal{F}' = \mathcal{F}_{|\Omega'}$ and $\mu' = \mu_{|\mathcal{F}'}$. For all maps $f : \Omega' \to [0, +\infty]$ (or

C), we define $\tilde{f} : \Omega \to [0, +\infty]$ (or **C**), by:

$$\tilde{f}(\omega) \triangleq \begin{cases} f(\omega) & \text{if} \quad \omega \in \Omega' \\ 0 & \text{if} \quad \omega \notin \Omega' \end{cases}$$

1. Show that $\mathcal{F}' \subseteq \mathcal{F}$ and conclude that μ' is therefore a well-defined measure on $(\Omega', \mathcal{F}')$.

2. Let $A \in \mathcal{F}'$ and $1'_A$ be the characteristic function of A defined on Ω'. Let 1_A be the characteristic function of A defined on Ω. Show that $\tilde{1}'_A = 1_A$.

3. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. Show that $\tilde{f} : (\Omega, \mathcal{F}) \to [0, +\infty]$ is also non-negative and measurable, and that we have:

$$\int_{\Omega'} f d\mu' = \int_{\Omega} \tilde{f} d\mu$$

4. Let $f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', \mu')$. Show that $\tilde{f} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and:

$$\int_{\Omega'} f d\mu' = \int_{\Omega} \tilde{f} d\mu$$

Definition 122 *$b : \mathbf{R}^+ \to \mathbf{C}$ is* **absolutely continuous***, if and only if b is right-continuous of finite variation, and b is absolutely continuous with respect to $a(t) = t$.*

Exercise 20. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map.

1. Show that b is absolutely continuous, if and only if there is $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ such that $b(t) = \int_0^t f(s)ds$, for all $t \in \mathbf{R}^+$.
2. Show that b absolutely continuous $\Rightarrow$ b continuous with $b(0) = 0$.

Exercise 21. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be an absolutely continuous map. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ be such that $b = f.t$. For all $n \geq 1$, we define

$f_n : \mathbf{R} \to \mathbf{C}$ by:

$$f_n(t) \stackrel{\triangle}{=} \begin{cases} f(t)1_{[0,n]}(t) & \text{if} \quad t \in \mathbf{R}^+ \\ 0 & \text{if} \quad t < 0 \end{cases}$$

1. Let $n \geq 1$. Show $f_n \in L^1_{\mathbf{C}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$ and for all $t \in [0, n]$:

$$b(t) = \int_0^t f_n dx$$

2. Show the existence of $N_n \in \mathcal{B}(\mathbf{R})$ such that $dx(N_n) = 0$, and for all $t \in N_n^c$, t is a Lebesgue point of f_n.

3. Show that for all $t \in \mathbf{R}$, and $\epsilon > 0$:

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} |f_n(s) - f_n(t)| ds \leq \frac{2}{dx(B(t,\epsilon))} \int_{B(t,\epsilon)} |f_n(s) - f_n(t)| ds$$

4. Show that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} f_n(s) ds = f_n(t)$$

5. Show similarly that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s) ds = f_n(t)$$

6. Show that for all $t \in N_n^c \cap [0, n[$, $b'(t)$ exists and $b'(t) = f(t)$.[1]

7. Show the existence of $N \in \mathcal{B}(\mathbf{R}^+)$, such that $dx(N) = 0$, and:

$$\forall t \in N^c \ , \ b'(t) \text{ exists with } b'(t) = f(t)$$

8. Conclude with the following:

[1] $b'(0)$ being a r.h.s derivative only.

Theorem 102 *A map* $b : \mathbf{R}^+ \to \mathbf{C}$ *is absolutely continuous, if and only if there exists* $f \in L^{1,loc}_{\mathbf{C}}(t)$ *such that:*

$$\forall t \in \mathbf{R}^+ \ , \ b(t) = \int_0^t f(s)ds$$

in which case, b *is almost surely differentiable with* $b' = f$ dx*-a.s.*