14. Maps of Finite Variation

Definition 108 *We call* **total variation** *of a map* $b : \mathbf{R}^+ \to \mathbf{C}$ *the map* $|b| : \mathbf{R}^+ \to [0, +\infty]$ *defined as:*

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) \stackrel{\triangle}{=} |b(0)| + \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$ *in* $[0, t]$, $n \geq 1$. *We say that* b *is of* **finite variation**, *if and only if:*

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) < +\infty$$

We say that b *is of* **bounded variation**, *if and only if:*

$$\sup_{t \in \mathbf{R}^+} |b|(t) < +\infty$$

Warning: The notation $|b|$ can be misleading: it can refer to the map $t \to |b(t)|$(the modulus), or to the map $t \to |b|(t)$ (the total variation).

Exercise 1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be non-decreasing with $a(0) \geq 0$.

1. Show that $|a| = a$ and a is of finite variation.
2. Show that the limit $\lim_{t\uparrow+\infty} a(t)$, denoted $a(\infty)$, exists in $\bar{\mathbf{R}}$.
3. Show that a is of bounded variation if and only if $a(\infty) < +\infty$.

Exercise 2. Let $b = b_1 + ib_2 : \mathbf{R}^+ \to \mathbf{C}$ be a map, b_1, b_2 real-valued.

1. Show that $|b_1| \leq |b|$ and $|b_2| \leq |b|$.
2. Show that $|b| \leq |b_1| + |b_2|$.
3. Show that b is of finite variation if and only if b_1, b_2 are.
4. Show that b is of bounded variation if and only if b_1, b_2 are.
5. Show that $|b|(0) = |b(0)|$.

Exercise 3. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be differentiable, such that b' is bounded on each compact interval of $\mathbf{R}^+$. Show that b is of finite variation.

Exercise 4. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is of class C^1, i.e. continuous and differentiable with continuous derivative, then b is of finite variation.

Exercise 5. Let $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, with $\int_0^t |f(s)|ds < +\infty$ for all $t \in \mathbf{R}^+$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ defined by:

$$\forall t \in \mathbf{R}^+ \ , \ b(t) \stackrel{\triangle}{=} \int_{\mathbf{R}^+} f 1_{[0,t]} ds$$

1. Show that b is of finite variation and:

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) \leq \int_0^t |f(s)|ds$$

2. Show that $f \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds) \Rightarrow b$ is of bounded variation.

Exercise 6. Show that if $b, b' : \mathbf{R}^+ \to \mathbf{C}$ are maps of finite variation, and $\alpha \in \mathbf{C}$, then $b + \alpha b'$ is also a map of finite variation. Prove the same result when the word 'finite' is replaced by 'bounded'.

Exercise 7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. For all $t \in \mathbf{R}^+$, let $\mathcal{S}(t)$ be the set of all finite subsets A of $[0, t]$, with $\mathrm{card}A \geq 2$. For all $A \in \mathcal{S}(t)$, we define:

$$S(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where it is understood that $t_0, \ldots, t_n$ are such that:

$$t_0 < t_1 < \ldots < t_n \text{ and } A = \{t_0, \ldots, t_n\} \subseteq [0, t]$$

1. Show that for all $t \in \mathbf{R}^+$, if $s_0 \leq \ldots \leq s_p$ $(p \geq 1)$ is a finite

sequence in $[0,t]$, then if:

$$S \stackrel{\triangle}{=} \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

either $S = 0$ or $S = S(A)$ for some $A \in \mathcal{S}(t)$.

2. Conclude that:

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) = |b(0)| + \sup\{S(A) : \ A \in \mathcal{S}(t)\}$$

3. Let $A \in \mathcal{S}(t)$ and $s \in [0,t]$. Show that $S(A) \leq S(A \cup \{s\})$.

4. Let $A, B \in \mathcal{S}(t)$. Show that:

$$A \subseteq B \ \Rightarrow \ S(A) \leq S(B)$$

5. Show that if $t_0 \leq \ldots \leq t_n$, $n \geq 1$, and $s_0 \leq \ldots \leq s_p$, $p \geq 1$, are finite sequences in $[0,t]$ such that:

$$\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$$

then:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

Exercise 8. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of finite variation. Let $s, t \in \mathbf{R}^+$, with $s \leq t$. We define:

$$\delta \stackrel{\triangle}{=} \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[s, t]$.

1. Let $s_0 \leq \ldots \leq s_p$ and $t_0 \leq \ldots \leq t_n$ be finite sequences in $[0, s]$ and $[s, t]$ respectively, where $n, p \geq 1$. Show that:

$$\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| + \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(t) - |b(0)|$$

2. Show that $\delta \leq |b|(t) - |b|(s)$.

3. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$, where $n \geq 1$, and suppose that $s = t_j$ for some $0 < j < n$. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(s) - |b(0)| + \delta \tag{1}$$

4. Show that inequality (1) holds, for all $t_0 \leq \ldots \leq t_n$ in $[0, t]$.

5. Prove the following:

Theorem 80 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Then, for all $s, t \in \mathbf{R}^+$, $s \leq t$, we have:*

$$|b|(t) - |b|(s) = \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[s, t]$.

Exercise 9. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Show that $|b|$ is non-decreasing with $|b|(0) \geq 0$, and $||b|| = |b|$.

Definition 109 *Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation. Let:*

$$\begin{aligned} |b|^+ &\stackrel{\triangle}{=} \frac{1}{2}(|b| + b) \\ |b|^- &\stackrel{\triangle}{=} \frac{1}{2}(|b| - b) \end{aligned}$$

$|b|^+$, $|b|^-$ are respectively the **positive**, **negative variation** *of b.*

Exercise 10. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation.

1. Show that $|b| = |b|^+ + |b|^-$ and $b = |b|^+ - |b|^-$.
2. Show that $|b|^+(0) = b^+(0) \geq 0$ and $|b|^-(0) = b^-(0) \geq 0$.
3. Show that for all $s, t \in \mathbf{R}^+$, $s \leq t$, we have:

$$|b(t) - b(s)| \leq |b|(t) - |b|(s)$$

4. Show that $|b|^+$ and $|b|^-$ are non-decreasing.

Exercise 11. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of finite variation. Show the existence of $b_1, b_2, b_3, b_4 : \mathbf{R}^+ \to \mathbf{R}^+$, non-decreasing with $b_i(0) \geq 0$, such that $b = b_1 - b_2 + i(b_3 - b_4)$. Show conversely that if $b : \mathbf{R}^+ \to \mathbf{C}$ is a map with such decomposition, then it is of finite variation.

Exercise 12. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation, and $x_0 \in \mathbf{R}^+$.

1. Show that the limit $|b|(x_0+) = \lim_{t \downarrow\downarrow x_0} |b|(t)$ exists in $\mathbf{R}$ and is equal to $\inf_{x_0 < t} |b|(t)$.

2. Show that $|b|(x_0) \leq |b|(x_0+)$.

3. Given $\epsilon > 0$, show the existence of $y_0 \in \mathbf{R}^+$, $x_0 < y_0$, such that:

$$
\begin{array}{lcl}
u \in]x_0, y_0] & \Rightarrow & |b(u) - b(x_0)| \leq \epsilon/2 \\
u \in]x_0, y_0] & \Rightarrow & |b|(y_0) - |b|(u) \leq \epsilon/2
\end{array}
$$

Exercise 13. Further to exercise (12), let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, y_0]$, for which there exists j with $0 < j < n-1$, $x_0 = t_j$ and $x_0 < t_{j+1}$.

1. Show that $\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)|$.
2. Show that $|b(t_{j+1}) - b(t_j)| \leq \epsilon/2$.
3. Show that $\sum_{i=j+2}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b|(t_{j+1}) \leq \epsilon/2$.
4. Show that for all finite sequences $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[0, y_0]$:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)| + \epsilon$$
5. Show that $|b|(y_0) \leq |b|(x_0) + \epsilon$.
6. Show that $|b|(x_0+) \leq |b|(x_0)$ and that $|b|$ is right-continuous.

Exercise 14. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a left-continuous map of finite variation, and let $x_0 \in \mathbf{R}^+ \setminus \{0\}$.

1. Show that the limit $|b|(x_0-) = \lim_{t \uparrow\uparrow x_0} |b|(t)$ exists in $\mathbf{R}$, and is equal to $\sup_{t<x_0} |b|(t)$.
2. Show that $|b|(x_0-) \leq |b|(x_0)$.
3. Given $\epsilon > 0$, show the existence of $y_0 \in [0, x_0[$, such that:

$$u \in [y_0, x_0[\quad \Rightarrow \quad |b(x_0) - b(u)| \leq \epsilon/2$$
$$u \in [y_0, x_0[\quad \Rightarrow \quad |b|(u) - |b|(y_0) \leq \epsilon/2$$

Exercise 15. Further to exercise (14), let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, x_0]$, such that $y_0 = t_j$ for some $0 < j < n-1$, and $x_0 = t_n$. We denote $k = \max\{i : \ j \leq i \ , \ t_i < x_0\}$.

1. Show that $j \leq k \leq n-1$ and $t_k \in [y_0, x_0[$.
2. Show that $\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)|$.
3. Show that $\sum_{i=j+1}^{k} |b(t_i) - b(t_{i-1})| \leq |b|(t_k) - |b|(y_0) \leq \epsilon/2$, where if $j = k$, the corresponding sum is zero.
4. Show that $\sum_{i=k+1}^{n} |b(t_i) - b(t_{i-1})| = |b(x_0) - b(t_k)| \leq \epsilon/2$.
5. Show that for all finite sequences $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[0, x_0]$:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)| + \epsilon$$
6. Show that $|b|(x_0) \leq |b|(y_0) + \epsilon$.
7. Show that $|b|(x_0) \leq |b|(x_0-)$ and that $|b|$ is left-continuous.
8. Prove the following:

Theorem 81 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Then:*

$$\begin{array}{rcl} b \text{ is right-continuous} & \Rightarrow & |b| \text{ is right-continuous} \\ b \text{ is left-continuous} & \Rightarrow & |b| \text{ is left-continuous} \\ b \text{ is continuous} & \Rightarrow & |b| \text{ is continuous} \end{array}$$

Exercise 16. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be an $\mathbf{R}$-valued map of finite variation.

1. Show that if b is right-continuous, then so are $|b|^+$ and $|b|^-$.
2. State and prove similar results for left-continuity and continuity.

Exercise 17. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Show the existence of $b_1, b_2, b_3, b_4 : \mathbf{R}^+ \to \mathbf{R}^+$, right-continuous and non-decreasing maps with $b_i(0) \geq 0$, such that:

$$b = b_1 - b_2 + i(b_3 - b_4)$$

Exercise 18. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map. Let $t \in \mathbf{R}^+$. For all $p \geq 1$, we define:

$$S_p \stackrel{\triangle}{=} |b(0)| + \sum_{k=1}^{2^p} |b(kt/2^p) - b((k-1)t/2^p)|$$

1. Show that for all $p \geq 1$, $S_p \leq S_{p+1}$ and define $S = \sup_{p\geq 1} S_p$.

2. Show that $S \leq |b|(t)$.

Exercise 19. Further to exercise (18), let $t_0 < \ldots < t_n$ be a finite sequence of distinct elements of $[0, t]$. Let $\epsilon > 0$.

1. Show that for all $i = 0, \ldots, n$, there exists $p_i \geq 1$ and $q_i \in \{0, 1, \ldots, 2^{p_i}\}$ such that:

$$0 \leq t_0 \leq \frac{q_0 t}{2^{p_0}} < t_1 \leq \frac{q_1 t}{2^{p_1}} < \ldots < t_n \leq \frac{q_n t}{2^{p_n}} \leq t$$

and:

$$|b(t_i) - b(q_i t/2^{p_i})| \leq \epsilon \ , \ \forall i = 0, \dots, n$$

2. Show the existence of $p \geq 1$, and $k_0, \dots, k_n \in \{0, \dots, 2^p\}$ with:

$$0 \leq t_0 \leq \frac{k_0 t}{2^p} < t_1 \leq \frac{k_1 t}{2^p} < \dots < t_n \leq \frac{k_n t}{2^p} \leq t$$

and:

$$|b(t_i) - b(k_i t/2^p)| \leq \epsilon \ , \ \forall i = 0, \dots, n$$

3. Show that:

$$\sum_{i=1}^{n} |b(k_i t/2^p) - b(k_{i-1} t/2^p)| \leq S_p - |b(0)|$$

4. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)| + 2n\epsilon$$

5. Show that:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)|$$
6. Conclude that $|b|(t) \leq S$.
7. Prove the following:

Theorem 82 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous or left-continuous. Then, for all $t \in \mathbf{R}^+$:*

$$|b|(t) = |b(0)| + \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(kt/2^n) - b((k-1)t/2^n)|$$

Exercise 20. Let $b : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $b = 1_{\mathbf{Q}^+}$. Show that:

$$+\infty = |b|(1) \neq \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(k/2^n) - b((k-1)/2^n)| = 0$$

EXERCISE 21. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation.

1. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Explain why $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.
2. Is this still true, if b is right-continuous of finite variation?
3. Show that $d|b_1|^+, d|b_1|^-, d|b_2|^+$ and $d|b_2|^-$ are finite measures.
4. Let $db = d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$. Show that db is a well-defined complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.
5. Show that $db(\{0\}) = b(0)$.
6. Show that for all $s, t \in \mathbf{R}^+$, $s \le t$, $db(]s,t]) = b(t) - b(s)$.
7. Show that $\lim_{t\to+\infty} b(t)$ exists in $\mathbf{C}$. We denote $b(\infty)$ this limit.
8. Show that $db(\mathbf{R}^+) = b(\infty)$.
9. Proving the uniqueness of db, justify the following:

Definition 110 *Let* $b : \mathbf{R}^+ \to \mathbf{C}$ *be a right-continuous map of bounded variation. There exists a unique complex measure* db *on* $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, *such that:*

$$\begin{array}{ll} (i) & db(\{0\}) = b(0) \\ (ii) & \forall s,t \in \mathbf{R}^+ \; s \leq t \;,\; db(]s,t]) = b(t) - b(s) \end{array}$$

db is called the **complex Stieltjes measure** *associated with b.*

EXERCISE 22. Show that if $a : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $a(0) \geq 0$ and $a(\infty) < +\infty$, then definition (110) of da coincides with the already known definition (24).

EXERCISE 23. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Explain why $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. Why is it in general impossible to define:

$$db \stackrel{\triangle}{=} d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$$

Warning: it does not make sense to write something like 'db', unless b is either right-continuous, non-decreasing and $b(0) \geq 0$, or b is a right-continuous map of bounded variation.

EXERCISE 24. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. For all $T \in \mathbf{R}^+$, we define $b^T : \mathbf{R}^+ \to \mathbf{C}$ as $b^T(t) = b(T \wedge t)$ for all $t \in \mathbf{R}^+$.

1. Show that for all $T \in \mathbf{R}^+$, $|b^T| = |b|^T$.

2. Show that if b is of finite variation, then for all $T \in \mathbf{R}^+$, b^T is of bounded variation, and we have $|b^T|(\infty) = |b|(T) < +\infty$.

3. Show that if b is right-continuous and of finite variation, for all $T \in \mathbf{R}^+$, db^T is the unique complex measure on $\mathbf{R}^+$, with:

$$(i) \qquad db^T(\{0\}) = b(0)$$

$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \le t\ ,\ db^T(]s,t]) = b(T\wedge t) - b(T\wedge s)$$

4. Show that if b is $\mathbf{R}$-valued and of finite variation, for all $T \in \mathbf{R}^+$, we have $|b^T|^+ = (|b|^+)^T$ and $|b^T|^- = (|b|^-)^T$.

5. Show that if b is right-continuous and of bounded variation, for all $T \in \mathbf{R}^+$, we have $db^T = db^{[0,T]} = db([0,T]\cap\cdot\,)$

6. Show that if b is right-continuous, non-decreasing with $b(0) \ge 0$, for all $T \in \mathbf{R}^+$, we have $db^T = db^{[0,T]} = db([0,T]\cap\cdot\,)$

Exercise 25. Let μ, ν be two finite measures on $\mathbf{R}^+$, such that:

$$(i) \qquad \mu(\{0\}) \le \nu(\{0\})$$

$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \le t\ ,\ \mu(]s,t]) \le \nu(]s,t])$$

We define $a, c : \mathbf{R}^+ \to \mathbf{R}^+$ by $a(t) = \mu([0,t])$ and $c(t) = \nu([0,t])$.

1. Show that a and c are right-continuous, non-decreasing with $a(0) \ge 0$ and $c(0) \ge 0$.

2. Show that $da = \mu$ and $dc = \nu$.

3. Show that $a \leq c$.

4. Define $b : \mathbf{R}^+ \to \mathbf{R}^+$ by $b = c - a$. Show that b is right-continuous, non-decreasing with $b(0) \geq 0$.

5. Show that $da + db = dc$.

6. Conclude with the following:

Theorem 83 *Let μ, ν be two finite measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ with:*

$$\begin{array}{ll} (i) & \mu(\{0\}) \leq \nu(\{0\}) \\ (ii) & \forall s,t \in \mathbf{R}^+, s \leq t\ ,\ \mu(]s,t]) \leq \nu(]s,t]) \end{array}$$

Then $\mu \leq \nu$, i.e. for all $B \in \mathcal{B}(\mathbf{R}^+)$, $\mu(B) \leq \nu(B)$.

Exercise 26. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation.

1. Show that $|db|(\{0\}) = |b(0)| = d|b|(\{0\})$.

2. Let $s, t \in \mathbf{R}^+$, $s \leq t$. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[s, t]$, $n \geq 1$. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |db|(]s, t])$$

3. Show that $|b|(t) - |b|(s) \leq |db|(]s, t])$.

4. Show that $d|b| \leq |db|$.

5. Show that $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|) \subseteq L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$.

6. Show that $\mathbf{R}^+$ is metrizable and strongly σ-compact.

7. Show that $C^c_{\mathbf{C}}(\mathbf{R}^+)$, $C^b_{\mathbf{C}}(\mathbf{R}^+)$ are dense in $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

8. Let $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Given $\epsilon > 0$, show the existence of $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^+)$ such that $\int |\phi - h||db| \leq \epsilon$.

9. Show that $|\int h db| \leq |\int \phi db| + \epsilon$.

10. Show that:

$$\left| \int |\phi| d|b| - \int |h| d|b| \right| \leq \int |\phi - h| d|b| \leq \int |\phi - h| |db|$$

11. Show that $\int |\phi| d|b| \leq \int |h| d|b| + \epsilon$.

12. For all $n \geq 1$, we define:

$$\phi_n \stackrel{\triangle}{=} \phi(0) 1_{\{0\}} + \sum_{k=0}^{n2^n - 1} \phi(k/2^n) 1_{]k/2^n, (k+1)/2^n]}$$

Show there is $M \in \mathbf{R}^+$, such that $|\phi_n(x)| \leq M$ for all x and n.

13. Using the continuity of ϕ, show that $\phi_n \to \phi$.

14. Show that $\lim \int \phi_n db = \int \phi db$.

15. Show that $\lim \int |\phi_n| d|b| = \int |\phi| d|b|$.

16. Show that for all $n \geq 1$:

$$\int \phi_n db = \phi(0)b(0) + \sum_{k=0}^{n2^n-1} \phi(k/2^n)(b((k+1)/2^n) - b(k/2^n))$$

17. Show that $|\int \phi_n db| \leq \int |\phi_n| d|b|$ for all $n \geq 1$.

18. Show that $|\int \phi db| \leq \int |\phi| d|b|$.

19. Show that $|\int h db| \leq \int |h| d|b| + 2\epsilon$.

20. Show that $|\int h db| \leq \int |h| d|b|$ for all $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

21. Let $B \in \mathcal{B}(\mathbf{R}^+)$ and $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ be such that $|h| = 1$ and $db = \int h|db|$. Show that $|db|(B) = \int_B \bar{h} db$.

22. Conclude that $|db| \leq d|b|$.

EXERCISE 27. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show that for all $T \in \mathbf{R}^+$, $|db^T| = d|b^T| = d|b|^T$.
2. Show that $d|b|^T = d|b|^{[0,T]} = d|b|([0,T] \cap \cdot\,)$, and conclude:

Theorem 84 *If $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation, the total variation of its associated complex Stieltjes measure, is equal to the Stieltjes measure associated with its total variation, i.e.*

$$|db| = d|b|$$

If $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation, then for all $T \in \mathbf{R}^+$, b^T defined by $b^T(t) = b(T \wedge t)$, is right-continuous of bounded variation, and we have $|db^T| = d|b|([0,T] \cap \cdot\,) = d|b|^T$.

Definition 111 *Let $b : \mathbf{R}^+ \to E$ be a map, where E is a Hausdorff topological space. We say that b is* **cadlag** *with respect to E, if and only if b is right-continuous, and the limit:*

$$b(t-) = \lim_{s \uparrow\uparrow t} b(s)$$

exists in E, for all $t \in \mathbf{R}^+ \setminus \{0\}$. In the case when $E = \mathbf{C}$ or $E = \mathbf{R}$, given b cadlag, we define $b(0-) = 0$, and for all $t \in \mathbf{R}^+$:

$$\Delta b(t) \stackrel{\triangle}{=} b(t) - b(t-)$$

EXERCISE 28. Let $b : \mathbf{R}^+ \to E$ be cadlag, where E is a Hausdorff topological space. Suppose b has values in $E' \subseteq E$.

1. Show that for all $t > 0$, the limit $b(t-)$ is unique.
2. Show that E' is Hausdorff.
3. Explain why b may not be cadlag with respect to E'.
4. Show that b is cadlag with respect to $\bar{E}'$.
5. Show that $b : \mathbf{R}^+ \to \mathbf{R}$ is cadlag $\Leftrightarrow$ it is cadlag w.r. to $\mathbf{C}$.

Exercise 29.

1. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is cadlag, then b is continuous with $b(0) = 0$ if and only if $\Delta b(t) = 0$ for all $t \in \mathbf{R}^+$.

2. Show that if $a : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $a(0) \geq 0$, then a is cadlag (w.r. to $\mathbf{R}$ and $\mathbf{R}^+$) with $\Delta a \geq 0$.

3. Show that any linear combination of cadlag maps is itself cadlag.

4. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is a right-continuous map of finite variation, then b is cadlag.

5. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Show that $da(\{t\}) = \Delta a(t)$ for all $t \in \mathbf{R}^+$.

6. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of bounded variation. Show that $db(\{t\}) = \Delta b(t)$ for all $t \in \mathbf{R}^+$.

7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Let $T \in \mathbf{R}^+$. Show that:

$$\forall t \in \mathbf{R}^+ \ , \ b^T(t-) = \left\{ \begin{array}{ccc} b(t-) & \text{if} & t \leq T \\ b(T) & \text{if} & T < t \end{array} \right.$$

Show that $\Delta b^T = (\Delta b)1_{[0,T]}$, and $db^T(\{t\}) = \Delta b(t)1_{[0,T]}(t)$.

Exercise 30. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a cadlag map and $T \in \mathbf{R}^+$.

1. Show that if $t \to b(t-)$ is not bounded on $[0,T]$, there exists a sequence $(t_n)_{n\geq 1}$ in $[0,T]$ such that $|b(t_n)| \to +\infty$.

2. Suppose from now on that b is not bounded on $[0,T]$. Show the existence of a sequence $(t_n)_{n\geq 1}$ in $[0,T]$, such that $t_n \to t$ for some $t \in [0,T]$, and $|b(t_n)| \to +\infty$.

3. Define $R = \{n \geq 1 : \ t \leq t_n\}$ and $L = \{n \geq 1 : \ t_n < t\}$. Show that R and L cannot be both finite.

4. Suppose that R is infinite. Show the existence of $n_1 \geq 1$, with:

$$t_{n_1} \in [t, t+1[\cap[0, T]$$

5. If R is infinite, show there is $n_1 < n_2 < \ldots$ such that:

$$t_{n_k} \in [t, t+\frac{1}{k}[\cap[0, T]\ ,\ \forall k \geq 1$$

6. Show that $|b(t_{n_k})| \not\to +\infty$.

7. Show that if L is infinite, then $t > 0$ and there is an increasing sequence $n_1 < n_2 < \ldots$, such that:

$$t_{n_k} \in]t-\frac{1}{k}, t[\cap[0, T]\ ,\ \forall k \geq 1$$

8. Show that: $|b(t_{n_k})| \not\to +\infty$.

9. Prove the following:

Theorem 85 *Let* $b : \mathbf{R}^+ \to \mathbf{C}$ *be a cadlag map. Let* $T \in \mathbf{R}^+$*. Then* b *and the map* $t \to b(t-)$ *are bounded on* $[0, T]$*, i.e. there exists* $M \in \mathbf{R}^+$ *such that:*

$$|b(t)| \vee |b(t-)| \leq M \ , \ \forall t \in [0, T]$$