5. Lebesgue Integration

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition 39 *Let $A \subseteq \Omega$. We call* **characteristic function** *of A, the map $1_A : \Omega \to \mathbf{R}$, defined by:*

$$\forall \omega \in \Omega \ , \ 1_A(\omega) \stackrel{\triangle}{=} \begin{cases} 1 & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A \end{cases}$$

EXERCISE 1. Given $A \subseteq \Omega$, show that $1_A : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable if and only if $A \in \mathcal{F}$.

Definition 40 *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that a map $s : \Omega \to \mathbf{R}^+$ is a* **simple function** *on $(\Omega, \mathcal{F})$, if and only if s is of the form :*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$ and $A_i \in \mathcal{F}$, for all $i = 1, \ldots, n$.

Exercise 2. Show that $s : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ is measurable, whenever s is a simple function on $(\Omega, \mathcal{F})$.

Exercise 3. Let s be a simple function on $(\Omega, \mathcal{F})$ with representation $s = \sum_{i=1}^n \alpha_i 1_{A_i}$. Consider the map $\phi : \Omega \to \{0,1\}^n$ defined by $\phi(\omega) = (1_{A_1}(\omega), \ldots, 1_{A_n}(\omega))$. For each $y \in s(\Omega)$, pick one $\omega_y \in \Omega$ such that $y = s(\omega_y)$. Consider the map $\psi : s(\Omega) \to \{0,1\}^n$ defined by $\psi(y) = \phi(\omega_y)$.

1. Show that ψ is injective, and that $s(\Omega)$ is a finite subset of $\mathbf{R}^+$.
2. Show that $s = \sum_{\alpha \in s(\Omega)} \alpha 1_{\{s=\alpha\}}$
3. Show that any simple function s can be represented as:

$$s = \sum_{i=1}^n \alpha_i 1_{A_i}$$

where $n \geq 1, \alpha_i \in \mathbf{R}^+, A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Definition 41 *Let $(\Omega, \mathcal{F})$ be a measurable space, and s be a simple function on $(\Omega, \mathcal{F})$. We call* **partition** *of the simple function s, any representation of the form:*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$, $A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Exercise 4. Let s be a simple function on $(\Omega, \mathcal{F})$ with two partitions:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i} = \sum_{j=1}^{m} \beta_j 1_{B_j}$$

1. Show that $s = \sum_{i,j} \alpha_i 1_{A_i \cap B_j}$ is a partition of s.
2. Recall the convention $0 \times (+\infty) = 0$ and $\alpha \times (+\infty) = +\infty$ if $\alpha > 0$. For all $a_1, \ldots, a_p$ in $[0, +\infty], p \geq 1$ and $x \in [0, +\infty]$, prove the distributive property: $x(a_1 + \ldots + a_p) = xa_1 + \ldots + xa_p$.

3. Show that $\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j)$.

4. Explain why the following definition is legitimate.

Definition 42 *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space, and* s *be a simple function on* $(\Omega, \mathcal{F})$*. We define the* **integral** *of* s *with respect to* μ*, as the sum, denoted* $I^{\mu}(s)$*, defined by:*

$$I^{\mu}(s) \stackrel{\triangle}{=} \sum_{i=1}^{n} \alpha_i \mu(A_i) \in [0, +\infty]$$

where $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ *is any partition of* s*.*

Exercise 5. Let s, t be two simple functions on $(\Omega, \mathcal{F})$ with partitions $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ and $t = \sum_{j=1}^{m} \beta_j 1_{B_j}$. Let $\alpha \in \mathbf{R}^+$.

1. Show that $s + t$ is a simple function on $(\Omega, \mathcal{F})$ with partition:

$$s + t = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) 1_{A_i \cap B_j}$$

2. Show that $I^\mu(s + t) = I^\mu(s) + I^\mu(t)$.

3. Show that αs is a simple function on $(\Omega, \mathcal{F})$.

4. Show that $I^\mu(\alpha s) = \alpha I^\mu(s)$.

5. Why is the notation $I^\mu(\alpha s)$ meaningless if $\alpha = +\infty$ or $\alpha < 0$.

6. Show that if $s \leq t$ then $I^\mu(s) \leq I^\mu(t)$.

Exercise 6. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map. For all $n \geq 1$, we define:

$$s_n \stackrel{\triangle}{=} \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} 1_{\{\frac{k}{2^n} \leq f < \frac{k+1}{2^n}\}} + n 1_{\{n \leq f\}} \tag{1}$$

1. Show that s_n is a simple function on $(\Omega, \mathcal{F})$, for all $n \geq 1$.
2. Show that equation (1) is a partition s_n, for all $n \geq 1$.
3. Show that $s_n \leq s_{n+1} \leq f$, for all $n \geq 1$.
4. Show that $s_n \uparrow f$ as $n \to +\infty$[1].

[1] i.e. for all $\omega \in \Omega$, the sequence $(s_n(\omega))_{n \geq 1}$ is non-decreasing and converges to $f(\omega) \in [0, +\infty]$.

Theorem 18 *Let* $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ *be a non-negative and measurable map, where* $(\Omega, \mathcal{F})$ *is a measurable space. There exists a sequence* $(s_n)_{n\geq 1}$ *of simple functions on* $(\Omega, \mathcal{F})$ *such that* $s_n \uparrow f$*.*

Definition 43 *Let* $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ *be a non-negative and measurable map, where* $(\Omega, \mathcal{F}, \mu)$ *is a measure space. We define the* **Lebesgue integral** *of* f *with respect to* μ*, denoted* $\int f d\mu$*, as:*

$$\int f d\mu \stackrel{\triangle}{=} \sup\{I^{\mu}(s) \ : \ s \text{ simple function on } (\Omega, \mathcal{F}) \ , \ s \leq f\}$$

where, given any simple function s *on* $(\Omega, \mathcal{F})$*,* $I^{\mu}(s)$ *denotes its integral with respect to* μ*.*

Exercise 7. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map.

1. Show that $\int f d\mu \in [0, +\infty]$.
2. Show that $\int f d\mu = I^{\mu}(f)$, whenever f is a simple function.

3. Show that $\int g d\mu \leq \int f d\mu$, whenever $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable map with $g \leq f$.

4. Show that $\int (cf) d\mu = c \int f d\mu$, if $0 < c < +\infty$. Explain why both integrals are well defined. Is the equality still true for $c = 0$.

5. For $n \geq 1$, put $A_n = \{f > 1/n\}$, and $s_n = (1/n) 1_{A_n}$. Show that s_n is a simple function on $(\Omega, \mathcal{F})$ with $s_n \leq f$. Show that $A_n \uparrow \{f > 0\}$.

6. Show that $\int f d\mu = 0 \ \Rightarrow \mu(\{f > 0\}) = 0$.

7. Show that if s is a simple function on $(\Omega, \mathcal{F})$ with $s \leq f$, then $\mu(\{f > 0\}) = 0$ implies $I^{\mu}(s) = 0$.

8. Show that $\int f d\mu = 0 \ \Leftrightarrow \ \mu(\{f > 0\}) = 0$.

9. Show that $\int (+\infty) f d\mu = (+\infty) \int f d\mu$. Explain why both integrals are well defined.

10. Show that $(+\infty)1_{\{f=+\infty\}} \leq f$ and:

$$\int (+\infty)1_{\{f=+\infty\}} d\mu = (+\infty)\mu(\{f = +\infty\})$$

11. Show that $\int f d\mu < +\infty \ \Rightarrow \mu(\{f = +\infty\}) = 0$.

12. Suppose that $\mu(\Omega) = +\infty$ and take $f = 1$. Show that the converse of the previous implication is not true.

Exercise 8. Let s be a simple function on $(\Omega, \mathcal{F})$. Let $A \in \mathcal{F}$.

1. Show that $s1_A$ is a simple function on $(\Omega, \mathcal{F})$.

2. Show that for any partition $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ of s, we have:

$$I^\mu(s1_A) = \sum_{i=1}^n \alpha_i \mu(A_i \cap A)$$

3. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = I^{\mu}(s1_A)$. Show that ν is a measure on $\mathcal{F}$.

4. Suppose $A_n \in \mathcal{F}, A_n \uparrow A$. Show that $I^{\mu}(s1_{A_n}) \uparrow I^{\mu}(s1_A)$.

Exercise 9. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$, such that $f_n \uparrow f$.

1. Recall what the notation $f_n \uparrow f$ means.

2. Explain why $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

3. Let $\alpha = \sup_{n\geq 1} \int f_n d\mu$. Show that $\int f_n d\mu \uparrow \alpha$.

4. Show that $\alpha \leq \int f d\mu$.

5. Let s be any simple function on $(\Omega, \mathcal{F})$ such that $s \leq f$. Let $c \in]0, 1[$. For $n \geq 1$, define $A_n = \{cs \leq f_n\}$. Show that $A_n \in \mathcal{F}$ and $A_n \uparrow \Omega$.

6. Show that $cI^\mu(s1_{A_n}) \leq \int f_n d\mu$, for all $n \geq 1$.

7. Show that $cI^\mu(s) \leq \alpha$.

8. Show that $I^\mu(s) \leq \alpha$.

9. Show that $\int f d\mu \leq \alpha$.

10. Conclude that $\int f_n d\mu \uparrow \int f d\mu$.

Theorem 19 (Monotone Convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$ such that $f_n \uparrow f$. Then $\int f_n d\mu \uparrow \int f d\mu$.*

Exercise 10. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $a, b \in [0, +\infty]$.

1. Show that if $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are two sequences of non-negative and measurable maps such that $f_n \uparrow f$ and $g_n \uparrow g$, then $f_n + g_n \uparrow f + g$.

2. Show that $\int (f+g)d\mu = \int f d\mu + \int g d\mu$.

3. Show that $\int (af+bg)d\mu = a\int f d\mu + b\int g d\mu$.

Exercise 11. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. Define $f = \sum_{n=1}^{+\infty} f_n$.

1. Explain why $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ is well defined, non-negative and measurable.

2. Show that $\int f d\mu = \sum_{n=1}^{+\infty} \int f_n d\mu$.

Definition 44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{P}(\omega)$ be a property depending on $\omega \in \Omega$. We say that the property $\mathcal{P}(\omega)$ holds* **μ-almost surely**, *and we write $\mathcal{P}(\omega)$ μ-a.s., if and only if:*

$$\exists N \in \mathcal{F} \ , \ \mu(N) = 0 \ , \ \forall \omega \in N^c, \mathcal{P}(\omega) \textit{ holds}$$

Exercise 12. Let $\mathcal{P}(\omega)$ be a property depending on $\omega \in \Omega$, such that $\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}$ is an element of the σ-algebra $\mathcal{F}$.

1. Show that $\mathcal{P}(\omega)$, μ-a.s. $\Leftrightarrow$ $\mu(\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}^c) = 0$.
2. Explain why in general, the right-hand side of this equivalence cannot be used to defined μ-almost sure properties.

Exercise 13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$. Show that $\mu(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$.

Exercise 14. Let $(f_n)_{n\geq 1}$ be a sequence of maps $f_n : \Omega \to [0,+\infty]$.

1. Translate formally the statement $f_n \uparrow f$ μ-a.s.

2. Translate formally $f_n \to f$ μ-a.s. and $\forall n, (f_n \leq f_{n+1}$ μ-a.s.)

3. Show that the statements 1. and 2. are equivalent.

Exercise 15. Suppose that $f, g : (\Omega, \mathcal{F}) \to [0,+\infty]$ are non-negative and measurable with $f = g$ μ-a.s.. Let $N \in \mathcal{F}$, $\mu(N) = 0$ such that $f = g$ on N^c. Explain why $\int f d\mu = \int (f 1_N) d\mu + \int (f 1_{N^c}) d\mu$, all integrals being well defined. Show that $\int f d\mu = \int g d\mu$.

Exercise 16. Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative and measurable maps and f is a non-negative and measurable map, such that $f_n \uparrow f$ μ-a.s.. Let $N \in \mathcal{F}$, $\mu(N) = 0$, such that $f_n \uparrow f$ on N^c. Define $\bar{f}_n = f_n 1_{N^c}$ and $\bar{f} = f 1_{N^c}$.

1. Explain why $\bar{f}$ and the $\bar{f}_n$'s are non-negative and measurable.

2. Show that $\bar{f}_n \uparrow \bar{f}$.

3. Show that $\int f_n d\mu \uparrow \int f d\mu$.

Exercise 17. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. For $n \geq 1$, we define $g_n = \inf_{k\geq n} f_k$.

1. Explain why the g_n's are non-negative and measurable.

2. Show that $g_n \uparrow \liminf f_n$.

3. Show that $\int g_n d\mu \leq \int f_n d\mu$, for all $n \geq 1$.

4. Show that if $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ are two sequences in $\bar{\mathbf{R}}$ with $u_n \leq v_n$ for all $n \geq 1$, then $\liminf u_n \leq \liminf v_n$.

5. Show that $\int (\liminf f_n) d\mu \leq \liminf \int f_n d\mu$, and recall why all integrals are well defined.

Theorem 20 (Fatou Lemma) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. Then:*

$$\int (\liminf_{n\to+\infty} f_n) d\mu \leq \liminf_{n\to+\infty} \int f_n d\mu$$

Exercise 18. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map. Let $A \in \mathcal{F}$.

1. Recall what is meant by the induced measure space $(A, \mathcal{F}_{|A}, \mu_{|A})$. Why is it important to have $A \in \mathcal{F}$. Show that the restriction of f to A, $f_{|A} : (A, \mathcal{F}_{|A}) \to [0, +\infty]$ is measurable.

2. We define the map $\mu^A : \mathcal{F} \to [0, +\infty]$ by $\mu^A(E) = \mu(A \cap E)$, for all $E \in \mathcal{F}$. Show that $(\Omega, \mathcal{F}, \mu^A)$ is a measure space.

3. Consider the equalities:

$$\int (f1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A} \tag{2}$$

For each of the above integrals, what is the underlying measure space on which the integral is considered. What is the map being integrated. Explain why each integral is well defined.

4. Show that in order to prove (2), it is sufficient to consider the case when f is a simple function on $(\Omega, \mathcal{F})$.

5. Show that in order to prove (2), it is sufficient to consider the case when f is of the form $f = 1_B$, for some $B \in \mathcal{F}$.

6. Show that (2) is indeed true.

Definition 45 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. let $A \in \mathcal{F}$. We call* **partial Lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \bullet)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 19. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu$, for all $A \in \mathcal{F}$.

1. Show that ν is a measure on $\mathcal{F}$.

2. Show that:

$$\int g d\nu = \int g f d\mu$$

Theorem 21 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu$, for all $A \in \mathcal{F}$. Then, ν is a measure on $\mathcal{F}$, and for all $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ non-negative and measurable, we have:*

$$\int g d\nu = \int g f d\mu$$

Definition 46 *The* **L^1-spaces** *on a measure space $(\Omega, \mathcal{F}, \mu)$, are:*

$$L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \int |f| d\mu < +\infty \right\}$$

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \int |f| d\mu < +\infty \right\}$$

Exercise 20. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map.

1. Explain why the integral $\int |f| d\mu$ makes sense.
2. Show that $f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable, if $f(\Omega) \subseteq \mathbf{R}$.
3. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.
4. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$
5. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
6. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.

Definition 47 *Let $u : \Omega \to \mathbf{R}$ be a real-valued function defined on a set Ω. We call* **positive part** *and* **negative part** *of u the maps u^+ and u^- respectively, defined as $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$.*

EXERCISE 21. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let $u = Re(f)$ and $v = Im(f)$.

1. Show that $u = u^+ - u^-$, $v = v^+ - v^-$, $f = u^+ - u^- + i(v^+ - v^-)$.

2. Show that $|u| = u^+ + u^-, |v| = v^+ + v^-$

3. Show that $u^+, u^-, v^+, v^-, |f|, u, v, |u|, |v|$ all lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

4. Explain why the integrals $\int u^+ d\mu, \int u^- d\mu, \int v^+ d\mu, \int v^- d\mu$ are all well defined.

5. We define the integral of f with respect to μ, denoted $\int f d\mu$, as $\int f d\mu = \int u^+ d\mu - \int u^- d\mu + i \left(\int v^+ d\mu - \int v^- d\mu\right)$. Explain why $\int f d\mu$ is a well defined complex number.

6. In the case when $f(\Omega) \subseteq \mathbf{C} \cap [0, +\infty] = \mathbf{R}^+$, explain why this new definition of the integral of f with respect to μ is consistent with the one already known (43) for non-negative and measurable maps.

7. Show that $\int f d\mu = \int u d\mu + i \int v d\mu$ and explain why all integrals involved are well defined.

Definition 48 *Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space. We define the* **Lebesgue integral** *of f with respect to μ, denoted $\int f d\mu$, as:*

$$\int f d\mu \stackrel{\triangle}{=} \int u^+ d\mu - \int u^- d\mu + i\left(\int v^+ d\mu - \int v^- d\mu\right)$$

EXERCISE 22. Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $A \in \mathcal{F}$.

1. Show that $f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.
2. Show that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)$.
3. Show that $f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$
4. Show that $\int (f1_A) d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$.
5. Show that 4. is:$\int_A u^+ d\mu - \int_A u^- d\mu + i\left(\int_A v^+ d\mu - \int_A v^- d\mu\right)$.

Definition 49 *Let* $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ *, where* $(\Omega, \mathcal{F}, \mu)$ *is a measure space. let* $A \in \mathcal{F}$*. We call* **partial Lebesgue integral** *of* f *with respect to* μ *over* A*, the integral denoted* $\int_A f d\mu$*, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A *is the measure on* $(\Omega, \mathcal{F})$*,* $\mu^A = \mu(A \cap \bullet)$*,* $f_{|A}$ *is the restriction of* f *to* A *and* $\mu_{|A}$ *is the restriction of* μ *to* $\mathcal{F}_{|A}$*, the trace of* $\mathcal{F}$ *on* A*.*

EXERCISE 23. Let $f, g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and let $h = f + g$

1. Show that:

$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- d\mu + \int f^+ d\mu + \int g^+ d\mu$$

2. Show that $\int h d\mu = \int f d\mu + \int g d\mu$.

3. Show that $\int (-f) d\mu = -\int f d\mu$

4. Show that if $\alpha \in \mathbf{R}$ then $\int (\alpha f) d\mu = \alpha \int f d\mu$.

5. Show that if $f \leq g$ then $\int f d\mu \leq \int g d\mu$

6. Show the following theorem.

Theorem 22 *For all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$, we have:*

$$\int (\alpha f + g) d\mu = \alpha \int f d\mu + \int g d\mu$$

Exercise 24. Let f, g be two maps, and $(f_n)_{n \geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, such that:

$$\begin{array}{rl} (i) & \forall \omega \in \Omega \;,\; \lim_{n \to +\infty} f_n(\omega) = f(\omega) \text{ in } \mathbf{C} \\ (ii) & \forall n \geq 1 \;,\; |f_n| \leq g \\ (iii) & g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \end{array}$$

Let $(u_n)_{n \geq 1}$ be an arbitrary sequence in $\bar{\mathbf{R}}$.

1. Show that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$.
2. For $n \geq 1$, define $h_n = 2g - |f_n - f|$. Explain why Fatou lemma (20) can be applied to the sequence $(h_n)_{n \geq 1}$.
3. Show that $\liminf(-u_n) = -\limsup u_n$.
4. Show that if $\alpha \in \mathbf{R}$, then $\liminf(\alpha + u_n) = \alpha + \liminf u_n$.
5. Show that $u_n \to 0$ as $n \to +\infty$ if and only if $\limsup |u_n| = 0$.
6. Show that $\int (2g) d\mu \leq \int (2g) d\mu - \limsup \int |f_n - f| d\mu$
7. Show that $\limsup \int |f_n - f| d\mu = 0$.
8. Conclude that $\int |f_n - f| d\mu \to 0$ as $n \to +\infty$.

Theorem 23 (Dominated Convergence) *Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ such that $f_n \to f$ in $\mathbf{C}$[2] . Suppose that there exists some $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $|f_n| \leq g$ for all $n \geq 1$. Then $f, f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$, and:*

$$\lim_{n\to+\infty} \int |f_n - f| d\mu = 0$$

EXERCISE 25. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and put $z = \int f d\mu$. Let $\alpha \in \mathbf{C}$, be such that $|\alpha| = 1$ and $\alpha z = |z|$. Put $u = Re(\alpha f)$.

1. Show that $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$
2. Show that $u \leq |f|$
3. Show that $|\int f d\mu| = \int (\alpha f) d\mu$.
4. Show that $\int (\alpha f) d\mu = \int u d\mu$.

[2] i.e. for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to $f(\omega) \in \mathbf{C}$

5. Prove the following theorem.

Theorem 24 *Let* $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ *where* $(\Omega, \mathcal{F}, \mu)$ *is a measure space. We have:*

$$\left|\int f d\mu\right| \leq \int |f| d\mu$$