# 4. Measurability

**Definition 25** *Let $A$ and $B$ be two sets, and $f : A \to B$ be a map. Given $A' \subseteq A$, we call* **direct image** *of $A'$ by $f$ the set denoted $f(A')$, and defined by $f(A') = \{f(x) \ : \ x \in A'\}$.*

**Definition 26** *Let $A$ and $B$ be two sets, and $f : A \to B$ be a map. Given $B' \subseteq B$, we call* **inverse image** *of $B'$ by $f$ the set denoted $f^{-1}(B')$, and defined by $f^{-1}(B') = \{x \ : \ x \in A \ , \ f(x) \in B'\}$.*

Exercise 1. Let $A$ and $B$ be two sets, and $f : A \to B$ be a bijection from $A$ to $B$. Let $A' \subseteq A$ and $B' \subseteq B$.

1. Explain why the notation $f^{-1}(B')$ is potentially ambiguous.
2. Show that the inverse image of $B'$ by $f$ is in fact equal to the direct image of $B'$ by $f^{-1}$.
3. Show that the direct image of $A'$ by $f$ is in fact equal to the inverse image of $A'$ by $f^{-1}$.

**Definition 27** *Let* $(\Omega, \mathcal{T})$ *and* $(S, \mathcal{T}_S)$ *be two topological spaces. A map* $f : \Omega \to S$ *is said to be* **continuous** *if and only if:*

$$\forall B \in \mathcal{T}_S \ , \ f^{-1}(B) \in \mathcal{T}$$

*In other words, if and only if the inverse image of any open set in* $S$ *is an open set in* $\Omega$*.*

We Write $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ *is continuous*, as a way of emphasizing the two topologies $\mathcal{T}$ and $\mathcal{T}_S$ with respect to which $f$ is continuous.

**Definition 28** *Let* $E$ *be a set. A map* $d : E \times E \to [0, +\infty[$ *is said to be a* **metric** *on* $E$*, if and only if:*

$$\begin{array}{ll} (i) & \forall x, y \in E \ , \ d(x, y) = 0 \ \Leftrightarrow \ x = y \\ (ii) & \forall x, y \in E \ , \ d(x, y) = d(y, x) \\ (iii) & \forall x, y, z \in E \ , \ d(x, y) \leq d(x, z) + d(z, y) \end{array}$$

**Definition 29** *A* **metric space** *is an ordered pair* $(E, d)$ *where* $E$ *is a set, and* $d$ *is a metric on* $E$*.*

**Definition 30** *Let* $(E, d)$ *be a metric space. For all* $x \in E$ *and* $\epsilon > 0$*, we define the so-called* **open ball** *in* $E$*:*

$$B(x, \epsilon) \stackrel{\triangle}{=} \{y \ : \ y \in E \ , \ d(x, y) < \epsilon\}$$

*We call* **metric topology** *on* $E$*, associated with* $d$*, the topology* $\mathcal{T}_E^d$ *defined by:*

$$\mathcal{T}_E^d \stackrel{\triangle}{=} \{U \subseteq E \ , \forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subseteq U\}$$

Exercise 2. Let $\mathcal{T}_E^d$ be the metric topology associated with $d$, where $(E, d)$ is a metric space.

1. Show that $\mathcal{T}_E^d$ is indeed a topology on $E$.

2. Given $x \in E$ and $\epsilon > 0$, show that $B(x, \epsilon)$ is an open set in $E$.

EXERCISE 3. Show that the usual topology on $\mathbf{R}$ is nothing but the metric topology associated with $d(x,y) = |x-y|$.

EXERCISE 4. Let $(E,d)$ and $(F,\delta)$ be two metric spaces. Show that a map $f : E \to F$ is continuous, if and only if for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x),f(y)) < \epsilon$$

**Definition 31** *Let $(\Omega,\mathcal{T})$ and $(S,\mathcal{T}_S)$ be two topological spaces. A map $f : \Omega \to S$ is said to be a* **homeomorphism**, *if and only if $f$ is a continuous bijection, such that $f^{-1}$ is also continuous.*

**Definition 32** *A topological space $(\Omega,\mathcal{T})$ is said to be* **metrizable**, *if and only if there exists a metric $d$ on $\Omega$, such that the associated metric topology coincides with $\mathcal{T}$, i.e. $\mathcal{T}^d_\Omega = \mathcal{T}$.*

**Definition 33** *Let $(E, d)$ be a metric space and $F \subseteq E$. We call* **induced metric** *on $F$, denoted $d_{|F}$, the restriction of the metric $d$ to $F \times F$, i.e. $d_{|F} = d_{|F \times F}$.*

EXERCISE 5. Let $(E, d)$ be a metric space and $F \subseteq E$. We define $\mathcal{T}_F = (\mathcal{T}_E^d)_{|F}$ as the topology on $F$ induced by the metric topology on $E$. Let $\mathcal{T}'_F = \mathcal{T}_F^{d_{|F}}$ be the metric topology on $F$ associated with the induced metric $d_{|F}$ on $F$.

1. Show that $\mathcal{T}_F \subseteq \mathcal{T}'_F$.
2. Given $A \in \mathcal{T}'_F$, show that $A = (\cup_{x \in A} B(x, \epsilon_x)) \cap F$ for some $\epsilon_x > 0$, $x \in A$, where $B(x, \epsilon_x)$ denotes the open ball in $E$.
3. Show that $\mathcal{T}'_F \subseteq \mathcal{T}_F$.

**Theorem 12** *Let $(E, d)$ be a metric space and $F \subseteq E$. Then, the topology on $F$ induced by the metric topology, is equal to the metric topology on $F$ associated with the induced metric, i.e. $(\mathcal{T}_E^d)_{|F} = \mathcal{T}_F^{d_{|F}}$.*

Exercise 6. Let $\phi : \mathbf{R} \to ]-1, 1[$ be the map defined by:

$$\forall x \in \mathbf{R} \quad , \quad \phi(x) \stackrel{\triangle}{=} \frac{x}{|x| + 1}$$

1. Show that $[-1, 0[$ is not open in $\mathbf{R}$.
2. Show that $[-1, 0[$ is open in $[-1, 1]$.
3. Show that $\phi$ is a homeomorphism between $\mathbf{R}$ and $]-1, 1[$.
4. Show that $\lim_{x \to +\infty} \phi(x) = 1$ and $\lim_{x \to -\infty} \phi(x) = -1$.

Exercise 7. Let $\bar{\mathbf{R}} = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$. Let $\phi$ be defined as in exercise (6), and $\bar{\phi} : \bar{\mathbf{R}} \to [-1,1]$ be the map defined by:

$$\bar{\phi}(x) = \left\{ \begin{array}{rcl} \phi(x) & \text{if} & x \in \mathbf{R} \\ 1 & \text{if} & x = +\infty \\ -1 & \text{if} & x = -\infty \end{array} \right.$$

Define:

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{U \subseteq \bar{\mathbf{R}} \ , \ \bar{\phi}(U) \text{ is open in } [-1,1]\}$$

1. Show that $\bar{\phi}$ is a bijection from $\bar{\mathbf{R}}$ to $[-1,1]$, and let $\bar{\psi} = \bar{\phi}^{-1}$.

2. Show that $\mathcal{T}_{\bar{\mathbf{R}}}$ is a topology on $\bar{\mathbf{R}}$.

3. Show that $\bar{\phi}$ is a homeomorphism between $\bar{\mathbf{R}}$ and $[-1,1]$.

4. Show that $[-\infty, 2[, \ ]3, +\infty], \ ]3, +\infty[$ are open in $\bar{\mathbf{R}}$.

5. Show that if $\phi' : \bar{\mathbf{R}} \to [-1,1]$ is an arbitrary homeomorphism, then $U \subseteq \bar{\mathbf{R}}$ is open, if and only if $\phi'(U)$ is open in $[-1,1]$.

**Definition 34** *The* **usual topology** *on* $\bar{\mathbf{R}}$ *is defined as:*

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{U \subseteq \bar{\mathbf{R}}\ ,\ \bar{\phi}(U) \text{ is open in } [-1,1]\}$$

*where* $\bar{\phi} : \bar{\mathbf{R}} \to [-1,1]$ *is defined by* $\bar{\phi}(-\infty) = -1$, $\bar{\phi}(+\infty) = 1$ *and:*

$$\forall x \in \mathbf{R} \quad , \quad \bar{\phi}(x) \stackrel{\triangle}{=} \frac{x}{|x|+1}$$

EXERCISE 8. Let $\phi$ and $\bar{\phi}$ be as in exercise (7). Define:

$$\mathcal{T}' \stackrel{\triangle}{=} (\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}} \stackrel{\triangle}{=} \{U \cap \mathbf{R}\ ,\ U \in \mathcal{T}_{\bar{\mathbf{R}}}\}$$

1. Recall why $\mathcal{T}'$ is a topology on $\mathbf{R}$.
2. Show that for all $U \subseteq \bar{\mathbf{R}}$, $\phi(U \cap \mathbf{R}) = \bar{\phi}(U) \cap ]-1,1[$.
3. Explain why if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, $\phi(U \cap \mathbf{R})$ is open in $]-1,1[$.
4. Show that $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$, (the usual topology on $\mathbf{R}$).

5. Let $U \in \mathcal{T}_{\mathbf{R}}$. Show that $\bar{\phi}(U)$ is open in $]-1,1[$ and $[-1,1]$.

6. Show that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

7. Show that $\mathcal{T}_{\mathbf{R}} = \mathcal{T}'$, i.e. that the usual topology on $\bar{\mathbf{R}}$ induces the usual topology on $\mathbf{R}$.

8. Show that $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}} = \{B \cap \mathbf{R}\ ,\ B \in \mathcal{B}(\bar{\mathbf{R}})\}$

Exercise 9. Let $d : \bar{\mathbf{R}} \times \bar{\mathbf{R}} \to [0,+\infty[$ be defined by:

$$\forall (x,y) \in \bar{\mathbf{R}} \times \bar{\mathbf{R}} \quad , \quad d(x,y) = |\phi(x) - \phi(y)|$$

where $\phi$ is an arbitrary homeomorphism from $\bar{\mathbf{R}}$ to $[-1,1]$.

1. Show that $d$ is a metric on $\bar{\mathbf{R}}$.

2. Show that if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, then $\phi(U)$ is open in $[-1,1]$

3. Show that for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $y \in \phi(U)$, there exists $\epsilon > 0$ such that:
$$\forall z \in [-1, 1]\ ,\ |z - y| < \epsilon \ \Rightarrow\ z \in \phi(U)$$

4. Show that $\mathcal{T}_{\bar{\mathbf{R}}} \subseteq \mathcal{T}^d_{\bar{\mathbf{R}}}$.

5. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$ and $x \in U$, there is $\epsilon > 0$ such that:
$$\forall y \in \bar{\mathbf{R}}\ ,\ |\phi(x) - \phi(y)| < \epsilon \ \Rightarrow\ y \in U$$

6. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$, $\phi(U)$ is open in $[-1, 1]$.

7. Show that $\mathcal{T}^d_{\bar{\mathbf{R}}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

8. Prove the following theorem.

**Theorem 13** *The topological space* $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ *is metrizable.*

**Definition 35** *Let $(\Omega, \mathcal{F})$ and $(S, \Sigma)$ be two measurable spaces. A map $f : \Omega \to S$ is said to be* **measurable** *with respect to $\mathcal{F}$ and $\Sigma$, if and only if:*

$$\forall B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F}$$

We Write $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ *is measurable*, as a way of emphasizing the two $\sigma$-algebras $\mathcal{F}$ and $\Sigma$ with respect to which $f$ is measurable.

EXERCISE 10. Let $(\Omega, \mathcal{F})$ and $(S, \Sigma)$ be two measurable spaces. Let $S'$ be a set and $f : \Omega \to S$ be a map such that $f(\Omega) \subseteq S' \subseteq S$. We define $\Sigma'$ as the trace of $\Sigma$ on $S'$, i.e. $\Sigma' = \Sigma_{|S'}$.

1. Show that for all $B \in \Sigma$, we have $f^{-1}(B) = f^{-1}(B \cap S')$
2. Show that $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if $f : (\Omega, \mathcal{F}) \to (S', \Sigma')$ is itself measurable.
3. Let $f : \Omega \to \mathbf{R}^+$. Show that the following are equivalent:

$$(i) \qquad f : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \text{ is measurable}$$

$$(ii) \qquad f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ is measurable}$$
$$(iii) \qquad f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable}$$

Exercise 11. Let $(\Omega, \mathcal{F})$, $(S, \Sigma)$, $(S_1, \Sigma_1)$ be three measurable spaces. let $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ and $g : (S, \Sigma) \to (S_1, \Sigma_1)$ be two measurable maps.

1. For all $B \subseteq S_1$, show that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$
2. Show that $g \circ f : (\Omega, \mathcal{F}) \to (S_1, \Sigma_1)$ is measurable.

Exercise 12. Let $(\Omega, \mathcal{F})$ and $(S, \Sigma)$ be two measurable spaces. Let $f : \Omega \to S$ be a map. We define:

$$\Gamma \stackrel{\triangle}{=} \{B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F}\}$$

1. Show that $f^{-1}(S) = \Omega$.

2. Show that for all $B \subseteq S$, $f^{-1}(B^c) = (f^{-1}(B))^c$.

3. Show that if $B_n \subseteq S, n \geq 1$, then $f^{-1}(\cup_{n=1}^{+\infty} B_n) = \cup_{n=1}^{+\infty} f^{-1}(B_n)$

4. Show that $\Gamma$ is a $\sigma$-algebra on $S$.

5. Prove the following theorem.

**Theorem 14** *Let $(\Omega, \mathcal{F})$ and $(S, \Sigma)$ be two measurable spaces, and $\mathcal{A}$ be a set of subsets of $S$ generating $\Sigma$, i.e. such that $\Sigma = \sigma(\mathcal{A})$. Then $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if:*

$$\forall B \in \mathcal{A} \quad , \quad f^{-1}(B) \in \mathcal{F}$$

**Exercise 13.** Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. Let $f : \Omega \to S$ be a map. Show that if $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ is continuous, then $f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))$ is measurable.

**Exercise 14.** We define the following subsets of the power set $\mathcal{P}(\bar{\mathbf{R}})$:

$$\begin{aligned}
\mathcal{C}_1 &\stackrel{\triangle}{=} \{[-\infty, c]\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_2 &\stackrel{\triangle}{=} \{[-\infty, c[\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_3 &\stackrel{\triangle}{=} \{[c, +\infty]\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_4 &\stackrel{\triangle}{=} \{]c, +\infty]\ ,\ c \in \mathbf{R}\}
\end{aligned}$$

1. Show that $\mathcal{C}_2$ and $\mathcal{C}_4$ are subsets of $\mathcal{T}_{\bar{\mathbf{R}}}$.
2. Show that the elements of $\mathcal{C}_1$ and $\mathcal{C}_3$ are closed in $\bar{\mathbf{R}}$.
3. Show that for all $i = 1, 2, 3, 4$, $\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\bar{\mathbf{R}})$.
4. Let $U$ be open in $\bar{\mathbf{R}}$. Explain why $U \cap \mathbf{R}$ is open in $\mathbf{R}$.

5. Show that any open subset of $\mathbf{R}$ is a countable union of open bounded intervals in $\mathbf{R}$.

6. Let $a < b$, $a, b \in \mathbf{R}$. Show that we have:

$$]a, b[ = \bigcup_{n=1}^{+\infty} ]a, b - 1/n] = \bigcup_{n=1}^{+\infty} [a + 1/n, b[$$

7. Show that for all $i = 1, 2, 3, 4$, $]a, b[ \in \sigma(\mathcal{C}_i)$.

8. Show that for all $i = 1, 2, 3, 4$, $\{\{-\infty\}, \{+\infty\}\} \subseteq \sigma(\mathcal{C}_i)$.

9. Show that for all $i = 1, 2, 3, 4$, $\sigma(\mathcal{C}_i) = \mathcal{B}(\bar{\mathbf{R}})$

10. Prove the following theorem.

**Theorem 15** *Let $(\Omega, \mathcal{F})$ be a measurable space, and $f : \Omega \to \bar{\mathbf{R}}$ be a map. The following are equivalent:*

$$\begin{array}{rl} (i) & f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable} \\ (ii) & \forall B \in \mathcal{B}(\bar{\mathbf{R}})\ ,\ \{f \in B\} \in \mathcal{F} \\ (iii) & \forall c \in \mathbf{R}\ ,\ \{f \leq c\} \in \mathcal{F} \\ (iv) & \forall c \in \mathbf{R}\ ,\ \{f < c\} \in \mathcal{F} \\ (v) & \forall c \in \mathbf{R}\ ,\ \{c \leq f\} \in \mathcal{F} \\ (vi) & \forall c \in \mathbf{R}\ ,\ \{c < f\} \in \mathcal{F} \end{array}$$

EXERCISE 15. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. Let $g$ and $h$ be the maps defined by $g(\omega) = \inf_{n\geq 1} f_n(\omega)$ and $h(\omega) = \sup_{n\geq 1} f_n(\omega)$, for all $\omega \in \Omega$.

1. Let $c \in \mathbf{R}$. Show that $\{c \leq g\} = \cap_{n=1}^{+\infty}\{c \leq f_n\}$.
2. Let $c \in \mathbf{R}$. Show that $\{h \leq c\} = \cap_{n=1}^{+\infty}\{f_n \leq c\}$.

3. Show that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

**Definition 36** *Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. We define:*

$$u \stackrel{\triangle}{=} \liminf_{n\to+\infty} v_n \stackrel{\triangle}{=} \sup_{n\geq 1}\left(\inf_{k\geq n} v_k\right)$$

*and:*

$$w \stackrel{\triangle}{=} \limsup_{n\to+\infty} v_n \stackrel{\triangle}{=} \inf_{n\geq 1}\left(\sup_{k\geq n} v_k\right)$$

*Then, $u, w \in \bar{\mathbf{R}}$ are respectively called* **lower limit** *and* **upper limit** *of the sequence $(v_n)_{n\geq 1}$.*

Exercise 16. Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. for $n \geq 1$ we define $u_n = \inf_{k\geq n} v_k$ and $w_n = \sup_{k\geq n} v_k$. Let $u$ and $w$ be the lower limit and upper limit of $(v_n)_{n\geq 1}$, respectively.

1. Show that $u_n \leq u_{n+1} \leq u$, for all $n \geq 1$.

2. Show that $w \leq w_{n+1} \leq w_n$, for all $n \geq 1$.

3. Show that $u_n \to u$ and $w_n \to w$ as $n \to +\infty$.

4. Show that $u_n \leq v_n \leq w_n$, for all $n \geq 1$.

5. Show that $u \leq w$.

6. Show that if $u = w$ then $(v_n)_{n\geq 1}$ converges to a limit $v \in \bar{\mathbf{R}}$, with $u = v = w$.

7. Show that if $a, b \in \mathbf{R}$ are such that $u < a < b < w$ then for all $n \geq 1$, there exist $N_1, N_2 \geq n$ such that $v_{N_1} < a < b < v_{N_2}$.

8. Show that if $a, b \in \mathbf{R}$ are such that $u < a < b < w$ then there exist two strictly increasing sequences of integers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that for all $k \geq 1$, we have $v_{n_k} < a < b < v_{m_k}$.

9. Show that if $(v_n)_{n\geq 1}$ converges to some $v \in \bar{\mathbf{R}}$, then $u = w$.

**Theorem 16** *Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. Then, the following are equivalent:*

$$(i) \qquad \liminf_{n\to+\infty} v_n = \limsup_{n\to+\infty} v_n$$

$$(ii) \qquad \lim_{n\to+\infty} v_n \quad \textit{exists in } \bar{\mathbf{R}}.$$

*in which case:*

$$\lim_{n\to+\infty} v_n = \liminf_{n\to+\infty} v_n = \limsup_{n\to+\infty} v_n$$

Exercise 17. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $\{f < g\} = \cup_{r\in\mathbf{Q}}(\{f < r\} \cap \{r < g\})$.
2. Show that the sets $\{f < g\}, \{f > g\}, \{f = g\}, \{f \leq g\}$, $\{f \geq g\}$ belong to the $\sigma$-algebra $\mathcal{F}$.

Exercise 18. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. We define $g = \liminf f_n$ and $h = \limsup f_n$ in the obvious way:

$$\forall \omega \in \Omega \ , \ g(\omega) \stackrel{\triangle}{=} \liminf_{n\to+\infty} f_n(\omega)$$

$$\forall \omega \in \Omega \ , \ h(\omega) \stackrel{\triangle}{=} \limsup_{n\to+\infty} f_n(\omega)$$

1. Show that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.
2. Show that $g \leq h$, i.e. $\forall \omega \in \Omega \ , \ g(\omega) \leq h(\omega)$.
3. Show that $\{g = h\} \in \mathcal{F}$.
4. Show that $\{\omega : \omega \in \Omega \ , \ \lim_{n\to+\infty} f_n(\omega) \text{ exists in } \bar{\mathbf{R}}\} \in \mathcal{F}$.
5. Suppose $\Omega = \{g = h\}$, and let $f(\omega) = \lim_{n\to+\infty} f_n(\omega)$, for all $\omega \in \Omega$. Show that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

**Exercise 19.** Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $-f, |f|, f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

2. Let $a \in \bar{\mathbf{R}}$. Explain why the map $a + f$ may not be well defined.

3. Show that $(a+f) : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, whenever $a \in \mathbf{R}$.

4. Show that $(a.f) : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $a \in \bar{\mathbf{R}}$. (Recall the convention $0.\infty = 0$).

5. Explain why the map $f + g$ may not be well defined.

6. Suppose that $f \geq 0$ and $g \geq 0$, i.e. $f(\Omega) \subseteq [0, +\infty]$ and also $g(\Omega) \subseteq [0, +\infty]$. Show that $\{f + g < c\} = \{f < c - g\}$, for all $c \in \mathbf{R}$. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

7. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable whenever $f + g$ is well-defined, i.e. when the following condition holds:

$$(\{f = +\infty\} \cap \{g = -\infty\}) \cup (\{f = -\infty\} \cap \{g = +\infty\}) = \emptyset$$

8. Show that $1/f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when $f(\Omega) \subseteq \mathbf{R} \setminus \{0\}$.

9. Suppose that $f$ is $\mathbf{R}$-valued. Show that $\bar{f}$ defined by $\bar{f}(\omega) = f(\omega)$ if $f(\omega) \neq 0$ and $\bar{f}(\omega) = 1$ if $f(\omega) = 0$, is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

10. Suppose $f$ and $g$ take values in $\mathbf{R}$. Let $\bar{f}$ be defined as in 9. Show that for all $c \in \mathbf{R}$, the set $\{fg < c\}$ can be expressed as:

$$(\{f > 0\} \cap \{g < c/\bar{f}\}) \uplus (\{f < 0\} \cap \{g > c/\bar{f}\}) \uplus (\{f = 0\} \cap \{f < c\})$$

11. Show that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when $f$ and $g$ take values in $\mathbf{R}$.

Exercise 20. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space. Let $\bar{f}, \bar{g}$, be defined by:

$$\bar{f}(\omega) \stackrel{\triangle}{=} \begin{cases} f(\omega) & \text{if} \quad f(\omega) \notin \{-\infty, +\infty\} \\ 1 & \text{if} \quad f(\omega) \in \{-\infty, +\infty\} \end{cases}$$

$\bar{g}(\omega)$ being defined in a similar way. Consider the partitions of $\Omega$, $\Omega = A_1 \uplus A_2 \uplus A_3 \uplus A_4 \uplus A_5$ and $\Omega = B_1 \uplus B_2 \uplus B_3 \uplus B_4 \uplus B_5$, where $A_1 = \{f \in ]0, +\infty[\}$, $A_2 = \{f \in ] - \infty, 0[\}$, $A_3 = \{f = 0\}$, $A_4 = \{f = -\infty\}$, $A_5 = \{f = +\infty\}$ and $B_1, B_2, B_3, B_4, B_5$ being defined in a similar way with $g$. Recall the conventions $0 \times (+\infty) = 0$, $(-\infty) \times (+\infty) = (-\infty)$, etc...

1. Show that $\bar{f}$ and $\bar{g}$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

2. Show that all $A_i$'s and $B_j$'s are elements of $\mathcal{F}$.

3. Show that for all $B \in \mathcal{B}(\bar{\mathbf{R}})$:

$$\{fg \in B\} = \biguplus_{i,j=1}^{5} (A_i \cap B_j \cap \{fg \in B\})$$

4. Show that $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j \cap \{\bar{f}\bar{g} \in B\}$, in the case when $1 \leq i \leq 3$ and $1 \leq j \leq 3$.

5. Show that $A_i \cap B_j \cap \{fg \in B\}$ is either equal to $\emptyset$ or $A_i \cap B_j$, in the case when $i \geq 4$ or $j \geq 4$.

6. Show that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

**Definition 37** *Let $(\Omega, \mathcal{T})$ be a topological space, and $A \subseteq \Omega$. We call* **closure** *of $A$ in $\Omega$, denoted $\bar{A}$, the set defined by:*

$$\bar{A} \stackrel{\triangle}{=} \{x \in \Omega \ : \ x \in U \in \mathcal{T} \Rightarrow U \cap A \neq \emptyset\}$$

**Exercise 21.** Let $(E, \mathcal{T})$ be a topological space, and $A \subseteq E$. Let $\bar{A}$ be the closure of $A$.

1. Show that $A \subseteq \bar{A}$ and that $\bar{A}$ is closed.
2. Show that if $B$ is closed and $A \subseteq B$, then $\bar{A} \subseteq B$.
3. Show that $\bar{A}$ is the smallest closed set in $E$ containing $A$.
4. Show that $A$ is closed if and only if $A = \bar{A}$.
5. Show that if $(E, \mathcal{T})$ is metrizable, then:

$$\bar{A} = \{x \in E \ : \ \forall \epsilon > 0 \ , \ B(x, \epsilon) \cap A \neq \emptyset\}$$

where $B(x, \epsilon)$ is relative to any metric $d$ such that $\mathcal{T}_E^d = \mathcal{T}$.

**Exercise 22.** Let $(E, d)$ be a metric space. Let $A \subseteq E$. For all $x \in E$, we define:

$$d(x, A) \stackrel{\triangle}{=} \inf\{d(x, y) \ : \ y \in A\} \stackrel{\triangle}{=} \Phi_A(x)$$

where it is understood that $\inf \emptyset = +\infty$.

1. Show that for all $x \in E$, $d(x, A) = d(x, \bar{A})$.
2. Show that $d(x, A) = 0$, if and only if $x \in \bar{A}$.
3. Show that for all $x, y \in E$, $d(x, A) \leq d(x, y) + d(y, A)$.
4. Show that if $A \neq \emptyset$, $|d(x, A) - d(y, A)| \leq d(x, y)$.
5. Show that $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is continuous.
6. Show that if $A$ is closed, then $A = \Phi_A^{-1}(\{0\})$

Exercise 23. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where $(E, d)$ is a metric space. We assume that for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to some $f(\omega) \in E$.

1. Explain why $\liminf f_n$ and $\limsup f_n$ may not be defined in an arbitrary metric space $E$.

2. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(A) \in \mathcal{F}$ for all closed subsets $A$ of $E$.

3. Show that for all $A$ closed in $E$, $f^{-1}(A) = (\Phi_A \circ f)^{-1}(\{0\})$, where the map $\Phi_A : E \to \bar{\mathbf{R}}$ is defined as in exercise (22).

4. Show that $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

5. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable.

**Theorem 17** *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where $(E, d)$ is a metric space. Then, if the limit $f = \lim f_n$ exists on $\Omega$, the map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is itself measurable.*

**Definition 38** *The* **usual topology** *on* $\mathbf{C}$*, the set of complex numbers, is defined as the metric topology associated with* $d(z,z') = |z-z'|$.

Exercise 24. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, where $(\Omega, \mathcal{F})$ is a measurable space. Let $u = Re(f)$ and $v = Im(f)$. Show that $u, v, |f| : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are all measurable.

Exercise 25. Define the subset of the power set $\mathcal{P}(\mathbf{C})$:

$$\mathcal{C} \stackrel{\triangle}{=} \{]a,b[\times]c,d[\ ,\ a,b,c,d \in \mathbf{R}\}$$

where it is understood that:

$$]a,b[\times]c,d[\stackrel{\triangle}{=} \{z = x+iy \in \mathbf{C}\ ,\ (x,y) \in ]a,b[\times]c,d[\}$$

1. Show that any element of $\mathcal{C}$ is open in $\mathbf{C}$.

2. Show that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{C})$.

3. Let $z = x+iy \in \mathbf{C}$. Show that if $|x| < \eta$ and $|y| < \eta$ then we have $|z| < \sqrt{2}\eta$.

4. Let $U$ be open in $\mathbf{C}$. Show that for all $z \in U$, there are rational numbers $a_z, b_z, c_z, d_z$ such that $z \in ]a_z, b_z[\times]c_z, d_z[\subseteq U$.

5. Show that $U$ can be written as $U = \cup_{n=1}^{+\infty} A_n$ where $A_n \in \mathcal{C}$.

6. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{C})$.

7. Let $(\Omega, \mathcal{F})$ be a measurable space, and $u, v : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be two measurable maps. Show that $u+iv : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.