12. Radon-Nikodym Theorem

In the following, $(\Omega, \mathcal{F})$ is an arbitrary measurable space.

Definition 96 *Let μ and ν be two (possibly complex) measures on $(\Omega, \mathcal{F})$. We say that ν is* **absolutely continuous** *with respect to μ, and we write $\nu << \mu$, if and only if, for all $E \in \mathcal{F}$:*

$$\mu(E) = 0 \ \Rightarrow \ \nu(E) = 0$$

Exercise 1. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Show that $\nu << \mu$ is equivalent to $|\nu| << \mu$.

Exercise 2. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Let $\epsilon > 0$. Suppose there exists a sequence $(E_n)_{n\geq 1}$ in $\mathcal{F}$ such that:

$$\forall n \geq 1 \ , \ \mu(E_n) \leq \frac{1}{2^n} \ , \ |\nu(E_n)| \geq \epsilon$$

Define:

$$E \stackrel{\triangle}{=} \limsup_{n\geq 1} E_n \stackrel{\triangle}{=} \bigcap_{n\geq 1} \bigcup_{k\geq n} E_k$$

1. Show that:

$$\mu(E) = \lim_{n\to+\infty} \mu\left(\bigcup_{k\geq n} E_k\right) = 0$$

2. Show that:

$$|\nu|(E) = \lim_{n\to+\infty} |\nu|\left(\bigcup_{k\geq n} E_k\right) \geq \epsilon$$

3. Let λ be a measure on $(\Omega, \mathcal{F})$. Can we conclude in general that:

$$\lambda(E) = \lim_{n\to+\infty} \lambda\left(\bigcup_{k\geq n} E_k\right)$$

4. Prove the following:

Theorem 58 *Let μ be a measure on $(\Omega, \mathcal{F})$ and ν be a complex measure on $(\Omega, \mathcal{F})$. The following are equivalent:*

$$\begin{array}{rl} (i) & \nu << \mu \\ (ii) & |\nu| << \mu \\ (iii) & \forall \epsilon > 0, \exists \delta > 0, \forall E \in \mathcal{F}, \mu(E) \leq \delta \Rightarrow |\nu(E)| < \epsilon \end{array}$$

EXERCISE 3. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$ such that $\nu << \mu$. Let $\nu_1 = Re(\nu)$ and $\nu_2 = Im(\nu)$.

1. Show that $\nu_1 << \mu$ and $\nu_2 << \mu$.
2. Show that $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^-$ are absolutely continuous w.r. to μ.

EXERCISE 4. Let μ be a finite measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed proper subset of $\mathbf{C}$. We assume that for all $E \in \mathcal{F}$

such that $\mu(E) > 0$, we have:

$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$

1. Show there is a sequence $(D_n)_{n\geq 1}$ of closed discs in $\mathbf{C}$, with:

$$S^c = \bigcup_{n=1}^{+\infty} D_n$$

   Let $\alpha_n \in \mathbf{C}$, $r_n > 0$ be such that $D_n = \{z \in \mathbf{C} : \ |z - \alpha_n| \leq r_n\}$.

2. Suppose $\mu(E_n) > 0$ for some $n \geq 1$, where $E_n = \{f \in D_n\}$. Show that:

$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f d\mu - \alpha_n \right| \leq \frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu \leq r_n$$

3. Show that for all $n \geq 1$, $\mu(\{f \in D_n\}) = 0$.

4. Prove the following:

Theorem 59 *Let μ be a finite measure on $(\Omega, \mathcal{F})$, $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed subset of $\mathbf{C}$ such that for all $E \in \mathcal{F}$ with $\mu(E) > 0$, we have:*

$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$

Then, $f \in S$ μ-a.s.

Exercise 5. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Let $(E_n)_{n \geq 1}$ be a sequence in $\mathcal{F}$ such that $E_n \uparrow \Omega$ and $\mu(E_n) < +\infty$ for all $n \geq 1$. Define $w : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ as:

$$w \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{1 + \mu(E_n)} 1_{E_n}$$

1. Show that for all $\omega \in \Omega$, $0 < w(\omega) \leq 1$.
2. Show that $w \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 6. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$ and ν be a finite measure on $(\Omega, \mathcal{F})$, such that $\nu << \mu$. Let $w \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ be such that $0 < w \leq 1$. We define $\bar{\mu} = \int w d\mu$, i.e.

$$\forall E \in \mathcal{F} \ , \ \bar{\mu}(E) \stackrel{\triangle}{=} \int_E w d\mu$$

1. Show that $\bar{\mu}$ is a finite measure on $(\Omega, \mathcal{F})$.

2. Show that $\phi = \nu + \bar{\mu}$ is also a finite measure on $(\Omega, \mathcal{F})$.

3. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, $fw \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and:

$$\int f d\phi = \int f d\nu + \int f w d\mu$$

4. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have:

$$\int |f| d\nu \leq \int |f| d\phi \leq \left(\int |f|^2 d\phi \right)^{\frac{1}{2}} (\phi(\Omega))^{\frac{1}{2}}$$

5. Show that $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, and for $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$:

$$\left| \int f d\nu \right| \leq \sqrt{\phi(\Omega)}.\|f\|_2$$

6. Show the existence of $g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ such that:

$$\forall f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \ , \ \int f d\nu = \int f g d\phi \tag{1}$$

7. Show that for all $E \in \mathcal{F}$ such that $\phi(E) > 0$, we have:

$$\frac{1}{\phi(E)} \int_E g d\phi \ \in \ [0, 1]$$

8. Show the existence of $g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ such that $g(\omega) \in [0, 1]$ for all $\omega \in \Omega$, and (1) still holds.

9. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have:

$$\int f(1-g)d\nu = \int fgwd\mu$$

10. Show that for all $n \geq 1$ and $E \in \mathcal{F}$,

$$f \stackrel{\triangle}{=} (1 + g + \ldots + g^n)1_E \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$$

11. Show that for all $n \geq 1$ and $E \in \mathcal{F}$,

$$\int_E (1 - g^{n+1})d\nu = \int_E g(1 + g + \ldots + g^n)wd\mu$$

12. Define:

$$h \stackrel{\triangle}{=} gw \left(\sum_{n=0}^{+\infty} g^n \right)$$

Show that if $A = \{0 \leq g < 1\}$, then for all $E \in \mathcal{F}$:

$$\nu(E \cap A) = \int_E h d\mu$$

13. Show that $\{h = +\infty\} = A^c$ and conclude that $\mu(A^c) = 0$.

14. Show that for all $E \in \mathcal{F}$, we have $\nu(E) = \int_E h d\mu$.

15. Show that if μ is σ-finite on $(\Omega, \mathcal{F})$, and ν is a finite measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$, there exists $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $h \geq 0$ and:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu$$

16. Prove the following:

Theorem 60 (Radon-Nikodym:1) *Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Let ν be a complex measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. Then, there exists some $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu$$

If ν is a signed measure on $(\Omega, \mathcal{F})$, we can assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. If ν is a finite measure on $(\Omega, \mathcal{F})$, we can assume $h \geq 0$.

Exercise 7. Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that:

$$\forall E \in \mathcal{F} \ , \ \int_E f d\mu = 0$$

where μ is a measure on $(\Omega, \mathcal{F})$.

1. Show that:

$$\int u^+ d\mu = \int_{\{u \geq 0\}} u d\mu$$

2. Show that $f = 0$ μ-a.s.

3. State and prove some uniqueness property in theorem (60).

Exercise 8. Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. Let $(E_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ such that $E_n \uparrow \Omega$ and $\nu(E_n) < +\infty$ for all $n \geq 1$. We define:

$$\forall n \geq 1\ ,\ \nu_n \stackrel{\triangle}{=} \nu^{E_n} \stackrel{\triangle}{=} \nu(E_n \cap \cdot)$$

1. Show that there exists $h_n \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ with $h_n \geq 0$ and:

$$\forall E \in \mathcal{F}\ ,\ \nu_n(E) = \int_E h_n d\mu \tag{2}$$

   for all $n \geq 1$.

2. Show that for all $E \in \mathcal{F}$,

$$\int_E h_n d\mu \leq \int_E h_{n+1} d\mu$$

3. Show that for all $n, p \geq 1$,

$$\mu(\{h_n - h_{n+1} > \frac{1}{p}\}) = 0$$

4. Show that $h_n \leq h_{n+1}$ μ-a.s.

5. Show the existence of a sequence $(h_n)_{n\geq 1}$ in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $0 \leq h_n \leq h_{n+1}$ for all $n \geq 1$ and with (2) still holding.

6. Let $h = \sup_{n\geq 1} h_n$. Show that:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu \tag{3}$$

7. Show that for all $n \geq 1$, $\int_{E_n} h d\mu < +\infty$.

8. Show that $h < +\infty$ μ-a.s.

9. Show there exists $h : (\Omega, \mathcal{F}) \to \mathbf{R}^+$ measurable, while (3) holds.

10. Show that for all $n \geq 1$, $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu^{E_n})$.

Theorem 61 (Radon-Nikodym:2) *Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. There exists a measurable map $h : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ such that:*

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu$$

Exercise 9. Let $h, h' : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. We assume:

$$\forall E \in \mathcal{F}\ ,\ \int_E h d\mu = \int_E h' d\mu$$

Let $(E_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ with $E_n \uparrow \Omega$ and $\mu(E_n) < +\infty$ for all $n \geq 1$. We define $F_n = E_n \cap \{h \leq n\}$ for all $n \geq 1$.

1. Show that for all n and $E \in \mathcal{F}$, $\int_E h d\mu^{F_n} = \int_E h' d\mu^{F_n} < +\infty$.

2. Show that for all $n, p \geq 1$, $\mu(F_n \cap \{h > h' + 1/p\}) = 0$.

3. Show that for all $n \geq 1$, $\mu(\{F_n \cap \{h \neq h'\}) = 0$.

4. Show that $\mu(\{h \neq h'\} \cap \{h < +\infty\}) = 0$.

5. Show that $h = h'$ μ-a.s.

6. State and prove some uniqueness property in theorem (61).

Exercise 10. Take $\Omega = \{*\}$ and $\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{*\}\}$. Let μ be the measure on $(\Omega, \mathcal{F})$ defined by $\mu(\emptyset) = 0$ and $\mu(\{*\}) = +\infty$. Let $h, h' : (\Omega, \mathcal{F}) \to [0, +\infty]$ be defined by $h(*) = 1 \neq 2 = h'(*)$. Show that we have:

$$\forall E \in \mathcal{F} \ , \ \int_E h d\mu = \int_E h' d\mu$$

Explain why this does not contradict the previous exercise.

Exercise 11. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that $\mu << |\mu|$.

2. Show the existence of some $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that:

$$\forall E \in \mathcal{F} \ , \ \mu(E) = \int_E h d|\mu|$$

3. If μ is a signed measure, can we assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$?

EXERCISE 12. Further to ex. (11), define $A_r = \{|h| < r\}$ for all $r > 0$.

1. Show that for all measurable partition $(E_n)_{n\geq 1}$ of A_r:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq r|\mu|(A_r)$$

2. Show that $|\mu|(A_r) = 0$ for all $0 < r < 1$.

3. Show that $|h| \geq 1$ $|\mu|$-a.s.

4. Suppose that $E \in \mathcal{F}$ is such that $|\mu|(E) > 0$. Show that:

$$\left| \frac{1}{|\mu|(E)} \int_E h d|\mu| \right| \leq 1$$

5. Show that $|h| \leq 1$ $|\mu|$-a.s.

6. Prove the following:

Theorem 62 *For all complex measure μ on $(\Omega, \mathcal{F})$, there exists h belonging to $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that $|h| = 1$ and:*

$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \int_E h d|\mu|$$

If μ is a signed measure on $(\Omega, \mathcal{F})$, we can assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$.

Exercise 13. Let $A \in \mathcal{F}$, and $(A_n)_{n \geq 1}$ be a sequence in $\mathcal{F}$.

1. Show that if $A_n \uparrow A$ then $1_{A_n} \uparrow 1_A$.

2. Show that if $A_n \downarrow A$ then $1_{A_n} \downarrow 1_A$.

3. Show that if $1_{A_n} \to 1_A$, then for all $\mu \in M^1(\Omega, \mathcal{F})$:

$$\mu(A) = \lim_{n \to +\infty} \mu(A_n)$$

EXERCISE 14. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $\nu = \int f d\mu \in M^1(\Omega, \mathcal{F})$.

2. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\nu|)$ be such that $|h| = 1$ and $\nu = \int h d|\nu|$. Show that for all $E, F \in \mathcal{F}$:

$$\int_E f 1_F d\mu = \int_E h 1_F d|\nu|$$

3. Show that if $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is bounded and measurable:

$$\forall E \in \mathcal{F} \ , \ \int_E fg d\mu = \int_E hg d|\nu|$$

4. Show that:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E f\bar{h} d\mu$$

5. Show that for all $E \in \mathcal{F}$,

$$\int_E Re(f\bar{h}) d\mu \geq 0 \quad , \quad \int_E Im(f\bar{h}) d\mu = 0$$

6. Show that $f\bar{h} \in \mathbf{R}^+$ μ-a.s.

7. Show that $f\bar{h} = |f|$ μ-a.s.

8. Prove the following:

Theorem 63 *Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, $\nu = \int f d\mu$ defined by:*

$$\forall E \in \mathcal{F}\ ,\ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

is a complex measure on $(\Omega, \mathcal{F})$ with total variation:

$$\forall E \in \mathcal{F}\ ,\ |\nu|(E) = \int_E |f| d\mu$$

EXERCISE 15. Let $\mu \in M^1(\Omega, \mathcal{F})$ be a signed measure. Suppose that $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int h d|\mu|$. Define $A = \{h = 1\}$ and $B = \{h = -1\}$.

1. Show that for all $E \in \mathcal{F}$, $\mu^+(E) = \int_E \frac{1}{2}(1 + h) d|\mu|$.

2. Show that for all $E \in \mathcal{F}$, $\mu^-(E) = \int_E \frac{1}{2}(1 - h) d|\mu|$.

3. Show that $\mu^+ = \mu^A = \mu(A \cap \cdot\,)$.

4. Show that $\mu^- = -\mu^B = -\mu(B \cap \cdot)$.

Theorem 64 (Hahn Decomposition) *Let μ be a signed measure on $(\Omega, \mathcal{F})$. There exist $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$, $\Omega = A \uplus B$ and for all $E \in \mathcal{F}$, $\mu^+(E) = \mu(A \cap E)$ and $\mu^-(E) = -\mu(B \cap E)$.*

Definition 97 *Let μ be a complex measure on $(\Omega, \mathcal{F})$. We define:*

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$$

and for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, the **Lebesgue integral** *of f with respect to μ, is defined as:*

$$\int f d\mu \stackrel{\triangle}{=} \int fhd|\mu|$$

where $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int hd|\mu|$.

Exercise 16. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that for all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable:

$$f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ \Leftrightarrow \ \int |f| d|\mu| < +\infty$$

2. Show that for $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $\int f d\mu$ is unambiguously defined.

3. Show that for all $E \in \mathcal{F}$, $1_E \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\int 1_E d\mu = \mu(E)$.

4. Show that if μ is a finite measure, then $|\mu| = \mu$.

5. Show that if μ is a finite measure, definition (97) of integral and space $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is consistent with that already known for measures.

6. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is a $\mathbf{C}$-vector space and that:

$$\int (f + \alpha g) d\mu = \int f d\mu + \alpha \int g d\mu$$

for all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$.

7. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$\left| \int f d\mu \right| \leq \int |f| d|\mu|$$

Exercise 17. Let $\mu, \nu \in M^1(\Omega, \mathcal{F})$, let $\alpha \in \mathbf{C}$.

1. Show that $|\alpha\nu| = |\alpha|.|\nu|$
2. Show that $|\mu + \nu| \leq |\mu| + |\nu|$
3. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu + \alpha\nu)$
4. Show that for all $E \in \mathcal{F}$:

$$\int 1_E d(\mu + \alpha\nu) = \int 1_E d\mu + \alpha \int 1_E d\nu$$

5. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$:

$$\int f d(\mu + \alpha\nu) = \int f d\mu + \alpha \int f d\nu$$

Exercise 18. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be non-negative and measurable. Let μ and ν be measures on $(\Omega, \mathcal{F})$, and $\alpha \in [0, +\infty]$:

1. Show that $\mu + \alpha\nu$ is a measure on $(\Omega, \mathcal{F})$ and:

$$\int f d(\mu + \alpha\nu) = \int f d\mu + \alpha \int f d\nu$$

2. Show that if $\mu \leq \nu$, then:

$$\int f d\mu \leq \int f d\nu$$

Exercise 19. Let $\mu \in M^1(\Omega, \mathcal{F})$, $\mu_1 = Re(\mu)$ and $\mu_2 = Im(\mu)$.

1. Show that $|\mu_1| \leq |\mu|$ and $|\mu_2| \leq |\mu|$.
2. Show that $|\mu| \leq |\mu_1| + |\mu_2|$.

3. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) = L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2)$.

4. Show that:

$$\begin{array}{rcl} L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1) & = & L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^-) \\ L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2) & = & L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^-) \end{array}$$

5. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$:

$$\int f d\mu = \int f d\mu_1^+ - \int f d\mu_1^- + i \left(\int f d\mu_2^+ - \int f d\mu_2^- \right)$$

Exercise 20. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $A \in \mathcal{F}$. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ be such that $|h| = 1$ and $\mu = \int h d|\mu|$. Recall that $\mu^A = \mu(A \cap \cdot\,)$ and $\mu_{|A} = \mu_{|(\mathcal{F}_{|A})}$ where $\mathcal{F}_{|A} = \{A \cap E\ ,\ E \in \mathcal{F}\} \subseteq \mathcal{F}$.

1. Show that we also have $\mathcal{F}_{|A} = \{E :\ E \in \mathcal{F}\ ,\ E \subseteq A\}$.

2. Show that $\mu^A \in M^1(\Omega, \mathcal{F})$ and $\mu_{|A} \in M^1(A, \mathcal{F}_{|A})$.

3. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show:

$$\sum_{n=1}^{+\infty} |\mu^A(E_n)| \leq |\mu|^A(E)$$

4. Show that we have $|\mu^A| \leq |\mu|^A$.

5. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of $A \cap E$. Show that:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu^A|(A \cap E)$$

6. Show that $|\mu^A|(A^c) = 0$.

7. Show that $|\mu^A| = |\mu|^A$.

8. Let $E \in \mathcal{F}_{|A}$ and $(E_n)_{n\geq 1}$ be an $\mathcal{F}_{|A}$-measurable partition of E.

Show that:

$$\sum_{n=1}^{+\infty} |\mu_{|A}(E_n)| \leq |\mu|_{|A}(E)$$

9. Show that $|\mu_{|A}| \leq |\mu|_{|A}$.

10. Let $E \in \mathcal{F}_{|A} \subseteq \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show that $(E_n)_{n\geq 1}$ is also an $\mathcal{F}_{|A}$-measurable partition of E, and conclude:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu_{|A}|(E)$$

11. Show that $|\mu_{|A}| = |\mu|_{|A}$.

12. Show that $\mu^A = \int h d|\mu^A|$.

13. Show that $h_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu_{|A}|)$ and $\mu_{|A} = \int h_{|A} d|\mu_{|A}|$.

14. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ , \ f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A) \ , \ f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$$

and:

$$\int f1_A d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$$

Definition 98 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. let $A \in \mathcal{F}$. We call* **partial Lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the complex measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \cdot)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 21. Prove the following:

Theorem 65 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. Then, $\nu = \int f d\mu$ defined as:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

is a complex measure on $(\Omega, \mathcal{F})$, with total variation:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E |f| d|\mu|$$

Moreover, for all measurable map $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have:

$$g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \ \Leftrightarrow \ gf \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$$

and when such condition is satisfied:

$$\int g d\nu = \int g f d\mu$$

Exercise 22. Let $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_n, \mathcal{F}_n)$ be measurable spaces, where $n \geq 2$. Let $\mu_1 \in M^1(\Omega_1, \mathcal{F}_1)$, ..., $\mu_n \in M^1(\Omega_n, \mathcal{F}_n)$. For all

$i \in \mathbf{N}_n$, let h_i belonging to $L^1_{\mathbf{C}}(\Omega_i, \mathcal{F}_i, |\mu_i|)$ be such that $|h_i| = 1$ and $\mu_i = \int h_i d|\mu_i|$. For all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we define:

$$\mu(E) \stackrel{\triangle}{=} \int_E h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

1. Show that $\mu \in M^1(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$

2. Show that for all measurable rectangle $A_1 \times \ldots \times A_n$:

$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

3. Prove the following:

Theorem 66 *Let $\mu_1, \ldots, \mu_n$ be n complex measures on measurable spaces $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_n, \mathcal{F}_n)$ respectively, where $n \geq 2$. There exists a unique complex measure $\mu_1 \otimes \ldots \otimes \mu_n$ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangle $A_1 \times \ldots \times A_n$, we have:*

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

Exercise 23. Further to theorem (66) and exercise (22):

1. Show that $|\mu_1 \otimes \ldots \otimes \mu_n| = |\mu_1| \otimes \ldots \otimes |\mu_n|$.

2. Show that $\|\mu_1 \otimes \ldots \otimes \mu_n\| = \|\mu_1\| \ldots \|\mu_n\|$.

3. Show that for all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$:
$$\mu_1 \otimes \ldots \otimes \mu_n(E) = \int_E h_1 \ldots h_n d|\mu_1 \otimes \ldots \otimes \mu_n|$$

4. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Show:
$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int f h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

5. let σ be a permutation of $\{1, \ldots, n\}$. Show that:
$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$